Chapter 1

Nuclear Fusion II

"The fusion rate is $n_1 n_2 \langle\sigma v\rangle$: density squared times a thermal average that hides the Gamow peak."

standard plasma-nuclear summary [6, 14]

Chapter **??** introduced fusion energetics and the Coulomb barrier. This chapter derives the reaction rate from first principles (beam–target and plasma), obtains $\sigma(v) \propto v^{-2}e^{-2G}$ at low energy, forms the thermal average $\langle\sigma v\rangle$, and identifies the Gamow peak that dominates thermonuclear burning. The discussion closes with the practical preference for D+T in laboratory fusion [6, 1, 14, 2].

1.1 Reaction rate from flux and cross section

1.1.1 Beam on a fixed target

Consider a beam of nuclei Y with number density n_Y and speed v incident on a target containing nuclei X. The fusion (or reaction) cross section σ is an effective area: if a Y trajectory would pierce the area σ centred on an X, a reaction occurs with the probability built into the definition of σ.

In a time $\mathrm{d}t$, each X is exposed to all Y particles that lie in a cylinder of base area σ and length $v\,\mathrm{d}t$. The volume of that cylinder is $\sigma v\,\mathrm{d}t$, so the number of Y nuclei in it is

$$n_Y\,\sigma\,v\,\mathrm{d}t. \tag{1.1}$$

The reaction rate *per target nucleus* X is therefore

$$\lambda_X = n_Y\,\sigma\,v. \tag{1.2}$$

If the number density of X is n_X, the number of reactions per unit volume per unit time is

$$R = n_X\,n_Y\,\sigma\,v. \tag{1.3}$$

This is the fundamental connection between microscopic cross section and macroscopic rate.

1.1.2 Thermonuclear plasma

In a fusion plasma both species move randomly; there is no laboratory "beam" and no fixed target. The same geometry applies in the *relative* frame: for any chosen X, construct the cylinder using the relative velocity v of approaching Y nuclei. Averaging over the thermal distribution of relative velocities replaces σv by its mean:

$$\boxed{R = n_X\, n_Y\, \langle \sigma v \rangle.} \tag{1.4}$$

If X and Y are identical species, the rate is $\frac{1}{2} n_X^2 \langle \sigma v \rangle$ (the $\frac{1}{2}$ avoids double counting pairs). Once a fusion event occurs the participating nuclei are consumed; the form of R is unchanged because σ already encodes the probability per encounter [6, 1, 14].

Key idea

Beam–target: $R = n_X n_Y \sigma v$ from the cylinder volume $\sigma v\, dt$. Plasma: $R = n_X n_Y \langle \sigma v \rangle$ with a Maxwellian average over relative velocity.

1.2 Thermal average $\langle \sigma v \rangle$

Relative velocities in a thermal plasma follow a Maxwell–Boltzmann distribution. Writing $f(v)\,\mathrm{d}v$ for the probability that the relative speed lies in $[v, v + \mathrm{d}v]$,

$$\langle \sigma v \rangle = \int_0^\infty \sigma(v)\, v\, f(v)\, \mathrm{d}v. \tag{1.5}$$

A standard form of the Maxwell distribution for relative speed (reduced mass μ) is

$$f(v)\,\mathrm{d}v = 4\pi v^2 \left(\frac{\mu}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{\mu v^2}{2 k_B T} \right) \mathrm{d}v. \tag{1.6}$$

It is often convenient to change variables to the relative kinetic energy

$$E = \frac{1}{2}\mu v^2, \qquad v\,\mathrm{d}v = \frac{\mathrm{d}E}{\mu}, \tag{1.7}$$

so that

$$\langle \sigma v \rangle \propto \int_0^\infty \sigma(E)\, E\, e^{-E/k_B T}\, \mathrm{d}E \tag{1.8}$$

(up to temperature-dependent normalisation factors from $f(v)$). The physics content is clear: $\langle \sigma v \rangle$ weights the energy-dependent reactivity $\sigma(E) v(E)$ by the thermal population $e^{-E/k_B T}$.

1.3 Low-energy fusion cross section

Two ingredients dominate $\sigma(v)$ at thermonuclear energies ($E \sim \mathrm{keV} \ll V_{\max}$):

1.3.1 Barrier penetration

The probability of tunneling through the Coulomb barrier is

$$P_{\text{pen}} \propto e^{-2G}, \tag{1.9}$$

where the Gamow factor G for a Coulomb barrier has the form

$$2G = \frac{2\pi Z_1 Z_2 e^2}{4\pi\varepsilon_0\,\hbar v} = \frac{2\pi Z_1 Z_2 e^2}{4\pi\varepsilon_0\,\hbar}\sqrt{\frac{\mu}{2E}}. \tag{1.10}$$

Equivalently one writes

$$G = \frac{a}{\sqrt{E}}, \qquad a = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0\,\hbar}\sqrt{\frac{\mu}{2}}\,\pi \quad \text{(up to conventional factors of order unity in definitions).} \tag{1.11}$$

As E increases, G falls and penetration becomes easier.

1.3.2 Geometrical / quantum factor $\propto 1/v^2$

At low energy the de Broglie wavelength $\lambda = h/(\mu v)$ is large compared with the nuclear radius R. A partial-wave argument makes this quantitative. Quantise the impact parameter by angular momentum $L = \ell\hbar = \mu v b_\ell$, so

$$b_\ell = \frac{\ell\hbar}{\mu v}. \tag{1.12}$$

The annular area between ℓ and $\ell + 1$ is

$$A_\ell = \pi b_{\ell+1}^2 - \pi b_\ell^2 = \frac{\pi\hbar^2}{\mu^2 v^2}(2\ell + 1). \tag{1.13}$$

Summing up to a maximum $\ell_{\max} \approx \mu v R/\hbar$ (impact parameters within the nuclear range) gives a total geometric factor

$$\sigma_{\text{geom}} \sim \pi\Big(R + \frac{\hbar}{\mu v}\Big)^2. \tag{1.14}$$

For thermonuclear energies, $\hbar/(\mu v) \gg R$ (lecture estimate: for $\sim 1\,\text{keV}$ deuterons, $\hbar/(\mu v) \sim 200\,\text{fm}$ while $R \sim 2\,\text{fm}$). The cross section is then dominated by the quantum term

$$\sigma \propto \frac{1}{v^2} \qquad \text{(equivalently } \sigma \propto 1/E\text{).} \tag{1.15}$$

1.3.3 Combined low-energy form

Multiplying the penetration probability by the $1/v^2$ factor,

$$\boxed{\sigma(v) \propto \frac{1}{v^2}\,e^{-2G} \qquad \text{with} \quad 2G \propto \frac{Z_1 Z_2}{v}.} \tag{1.16}$$

In energy variables the standard nuclear-astrophysics parametrisation is

$$\sigma(E) = \frac{S(E)}{E}\,e^{-b/\sqrt{E}}, \tag{1.17}$$

where $S(E)$ is the slowly varying **astrophysical S-factor** and b is the Gamow constant fixed by Z_1, Z_2, and μ [6, 14, 1].

Remark

The $1/v^2$ enhancement at low energy does *not* overcome the exponential Gamow suppression: overall $\sigma(E)$ still plummets as $E \to 0$. It does, however, matter for the precise shape of $\langle \sigma v \rangle$.

1.4 The Gamow peak

Insert $\sigma(E) \propto E^{-1} e^{-b/\sqrt{E}}$ into the thermal average. The energy integrand that governs $\langle \sigma v \rangle$ behaves as

$$I(E) \propto e^{-E/k_BT}\, e^{-b/\sqrt{E}} = \exp\!\left(-\frac{E}{k_BT} - \frac{b}{\sqrt{E}}\right). \tag{1.18}$$

Two competing exponentials appear:

- e^{-E/k_BT} falls as E rises (fewer hot particles);
- $e^{-b/\sqrt{E}}$ rises as E rises (easier tunneling).

Their product has a single maximum: the **Gamow peak**. Locating the maximum by $\mathrm{d}/\mathrm{d}E\,[\cdots] = 0$,

$$-\frac{1}{k_BT} + \frac{b}{2}E^{-3/2} = 0 \quad \Rightarrow \quad E_0 = \left(\frac{bk_BT}{2}\right)^{2/3}. \tag{1.19}$$

The peak energy E_0 lies well above k_BT but still far below $V_{\max}$ in typical stellar conditions. Almost all thermonuclear reactions occur in a narrow window around E_0; the wings of $I(E)$ contribute little.

Key idea

Gamow peak: maximum of $e^{-E/k_BT}e^{-b/\sqrt{E}}$ at $E_0 \propto (bk_BT)^{2/3}$. Fusion rates are exponentially sensitive to T through this window.

As temperature increases, E_0 shifts upward, the peak height grows rapidly, and $\langle \sigma v \rangle$ rises by many orders of magnitude. At extremely high T the simple low-energy form of σ eventually fails and other physics (resonances, different partial waves) enters; for solar and most laboratory fusion discussions the Gamow picture is the right first tool [14, 6, 2].

1.5 Putting it together: plasma reactivity

The volumetric reaction rate and power density are

$$R = n_X n_Y \langle \sigma v \rangle (T), \tag{1.20}$$

$$P = R\,Q = n_X n_Y \langle \sigma v \rangle\, Q. \tag{1.21}$$

Design of a fusion reactor therefore requires:

(i) high density product $n_X n_Y$ (or n^2 for a single species);

(ii) high temperature so that $\langle\sigma v\rangle$ is appreciable;

(iii) sufficient confinement time for the plasma to react before cooling or disassembling (Lawson criterion: $n\tau_E$ above a threshold depending on T and the fuel cycle).

1.5.1 Why D + T?

Among light-ion reactions, deuterium–tritium fusion

$$^2\mathrm{H} + {}^3\mathrm{H} \;\to\; {}^4\mathrm{He} + n + 17.6\,\mathrm{MeV} \tag{1.22}$$

has the largest $\langle\sigma v\rangle$ at the lowest temperature of any practical fuel cycle: $Z_1 Z_2 = 1$ (same as *pp* or DD) but a more favourable nuclear S-factor and Q. Coulomb barriers for D + ^{3}He or pure proton fuels are higher or the weak interaction intervenes (*pp*), making them harder for terrestrial reactors. Stellar cores burn hydrogen via the slow *pp* chain precisely because weak interactions set the pace; stars have billions of years [6, 1, 14].

Caution

High T alone is not enough: without density and confinement, a hot plasma radiates and expands before fusing. The triple product $nT\tau_E$ is the usual figure of merit for ignition studies.

1.6 Summary of the fusion rate theory

Collecting the chain of reasoning:

(1) Geometry: each X sweeps a volume $\sigma v\,\mathrm{d}t$ relative to the Y gas $\Rightarrow R = n_X n_Y \sigma v$.

(2) Thermal average: replace σv by $\langle\sigma v\rangle = \int \sigma(v) v f(v)\,\mathrm{d}v$.

(3) Cross section at low energy: $\sigma \propto v^{-2} e^{-2G}$ with $G \propto Z_1 Z_2 / v$.

(4) Maxwell × Gamow ⇒ Gamow peak at $E_0 \propto T^{2/3}$.

(5) Power density $P = n_X n_Y \langle\sigma v\rangle Q$; D + T maximises reactivity for laboratory T.

Takeaway

- $R = n_X n_Y \langle\sigma v\rangle$ from flux × cross section.
- $\langle\sigma v\rangle = \int \sigma(v) v f(v)\,\mathrm{d}v$ (Maxwellian).
- Barrier penetration e^{-2G}, $2G = 2\pi Z_1 Z_2 e^2/(4\pi\varepsilon_0 \hbar v)$.
- Low-energy $\sigma \propto v^{-2} e^{-2G}$ (partial-wave $1/v^2$ plus Gamow).
- Gamow peak of $e^{-E/kT} e^{-b/\sqrt{E}}$ sets the thermonuclear rate.
- D + T is the most accessible laboratory fuel cycle.

1.7 Further physics from the literature

Astrophysical S-factor. Writing $\sigma(E) = S(E)E^{-1}e^{-b/\sqrt{E}}$ removes the dominant Coulomb energy dependence so that $S(E)$ is nearly constant or slowly varying. Extrapolating $S(E)$ from laboratory energies (hundreds of keV) down to the Gamow window (tens of keV in the Sun) is a central task of nuclear astrophysics [14, 6].

Resonances. When a compound nuclear level sits in the Gamow window, $\sigma(E)$ can be resonantly enhanced (Breit–Wigner). The 3α process that forms carbon in red giants is a classic example (Hoyle state in ^{12}C) [14, 2].

Lawson criterion and ignition. A crude balance of fusion power against losses leads to a minimum $n\tau_E$ (or $nT\tau_E$) for net energy gain. Different fuel cycles and confinement schemes shift the required temperature and the peak of $\langle\sigma v\rangle$ [1, 6].

Neutron budgets in D + T. The 14 MeV neutron must be moderated and captured (often on lithium) to breed tritium if a closed fuel cycle is desired. Neutron damage to structural materials is a major engineering challenge [1, 2].

Reading guide

Basdevant [14] (Gamow peak and stellar rates); Krane [6] (cross sections and reactors); Dunlap [1] (plasma fusion concepts); Demtröder [2] (broader survey). Nuclear data for light-ion reactions: evaluated libraries and ENSDF/NuDat for residual nuclei [20, 21].

1.8 Problems with solutions

Problem 26.1 — Derive $R = n_X n_Y \sigma v$

A beam of Y nuclei with density n_Y and speed v strikes a medium with target density n_X. Using the cylinder of area σ and length $v\,dt$, derive the volumetric reaction rate R [6, 14].

Solution 26.1

Step 1. Volume swept relative to one X in time dt: $\sigma v\,dt$.
Step 2. Number of Y in that volume: $n_Y \sigma v\,dt$. Rate per X: $\lambda = n_Y \sigma v$.
Step 3. Per unit volume there are n_X targets, so

$$R = n_X n_Y \sigma v.$$

In a plasma, average over relative velocities: $R = n_X n_Y \langle\sigma v\rangle$.

Problem 26.2 — Annular partial-wave area

Show that the area between impact parameters $b_\ell = \ell\hbar/(\mu v)$ and $b_{\ell+1}$ equals $\pi\hbar^2(2\ell + 1)/(\mu^2 v^2)$. Hence argue $\sigma \propto 1/v^2$ when $\hbar/(\mu v) \gg R$.

Solution 26.2

Step 1.

$$A_\ell = \pi\Big(\frac{(\ell+1)\hbar}{\mu v}\Big)^2 - \pi\Big(\frac{\ell\hbar}{\mu v}\Big)^2 = \frac{\pi\hbar^2}{\mu^2 v^2}\left[(\ell+1)^2 - \ell^2\right] = \frac{\pi\hbar^2}{\mu^2 v^2}(2\ell+1).$$

Step 2. Summing $\ell = 0\ldots\ell_{\max}$ with $\ell_{\max} \sim \mu v R/\hbar$ gives $\sigma \sim \pi(R+\hbar/(\mu v))^2$. If $\hbar/(\mu v) \gg R$, $\sigma \sim \pi\hbar^2/(\mu^2 v^2) \propto 1/v^2$ [14, 6].

Problem 26.3 — Gamow exponent scaling

From $2G \propto Z_1 Z_2/v$ and $E = \frac{1}{2}\mu v^2$, show $2G \propto Z_1 Z_2\sqrt{\mu/E}$. How does increasing Z at fixed E affect the tunneling probability?

Solution 26.3

Step 1. $v = \sqrt{2E/\mu}$, so

$$\frac{1}{v} = \sqrt{\frac{\mu}{2E}} \quad \Rightarrow \quad 2G \propto Z_1 Z_2\sqrt{\frac{\mu}{E}}.$$

Step 2. Larger $Z_1 Z_2$ increases G and multiplies $P \sim e^{-2G}$ by an exponentially smaller factor. Heavy-ion fusion at thermonuclear energies is vastly harder than hydrogen fusion [6, 1].

Problem 26.4 — Location of the Gamow peak

Starting from $I(E) = \exp(-E/k_B T - b/\sqrt{E})$, derive $E_0 = (b\,k_B T/2)^{2/3}$ by maximising $I(E)$.

Solution 26.4

Step 1. Maximise the exponent $\phi(E) = -E/k_B T - bE^{-1/2}$:

$$\frac{\mathrm{d}\phi}{\mathrm{d}E} = -\frac{1}{k_B T} + \frac{b}{2}E^{-3/2} = 0.$$

Step 2.

$$E^{3/2} = \frac{b k_B T}{2} \quad \Rightarrow \quad E_0 = \left(\frac{b k_B T}{2}\right)^{2/3}.$$

This is the Gamow peak energy [14, 2].

Problem 26.5 — Rate scaling with density

A plasma of identical fuel nuclei has density n and reactivity $\langle\sigma v\rangle$. Write R including the identical-particle factor $\frac{1}{2}$. If n is doubled at fixed T, by what factor does R change?

Solution 26.5

$$R = \frac{1}{2} n^2 \langle \sigma v \rangle.$$

Doubling n multiplies R by 4. Fusion power density is quadratic in density at fixed temperature [6, 1].

Problem 26.6 — D + T energetics

In $\mathrm{D} + \mathrm{T} \to {}^4\mathrm{He} + n + 17.6\,\mathrm{MeV}$, the α and neutron share momentum. Show that the neutron carries about 14.1 MeV and the α about 3.5 MeV (nonrelativistic two-body kinematics in the CM frame).

Solution 26.6

Step 1. In the CM frame, $p_\alpha = p_n = p$ and

$$\frac{p^2}{2m_\alpha} + \frac{p^2}{2m_n} = Q.$$

Step 2. With $m_\alpha \approx 4m_n$,

$$K_n = \frac{m_\alpha}{m_\alpha + m_n} Q = \frac{4}{5} Q = 0.8 \times 17.6 = 14.08\,\mathrm{MeV},$$

$$K_\alpha = \frac{m_n}{m_\alpha + m_n} Q = \frac{1}{5} Q = 3.52\,\mathrm{MeV}.$$

The neutron carries $\sim 80\%$ of Q; in magnetic fusion it escapes the confining field and heats a blanket, while the charged α can heat the plasma [6, 14].

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