Chapter 1

Nuclear Fusion I

"Fusion powers the stars: light nuclei climb the binding curve toward iron, releasing energy, if they can tunnel the Coulomb barrier."

standard astrophysical summary [6, 14]

Fission exploits the decline of B/A for heavy nuclei. Fusion exploits the *rise* of B/A for light nuclei. This chapter reads the binding-energy curve as a map of energy release, derives the fusion Q-value from binding energies with a full neon–neon example, introduces the Coulomb barrier $V_{\max} = Z_1 Z_2 e^2/[4\pi\varepsilon_0(r_1 + r_2)]$, evaluates barrier heights for $\mathrm{Ne} + \mathrm{Ne}$ and $p + p$, and explains why stellar fusion proceeds by quantum tunneling at thermal energies far below the barrier [6, 1, 14, 2].

1.1 The B/A curve: fission on the right, fusion on the left

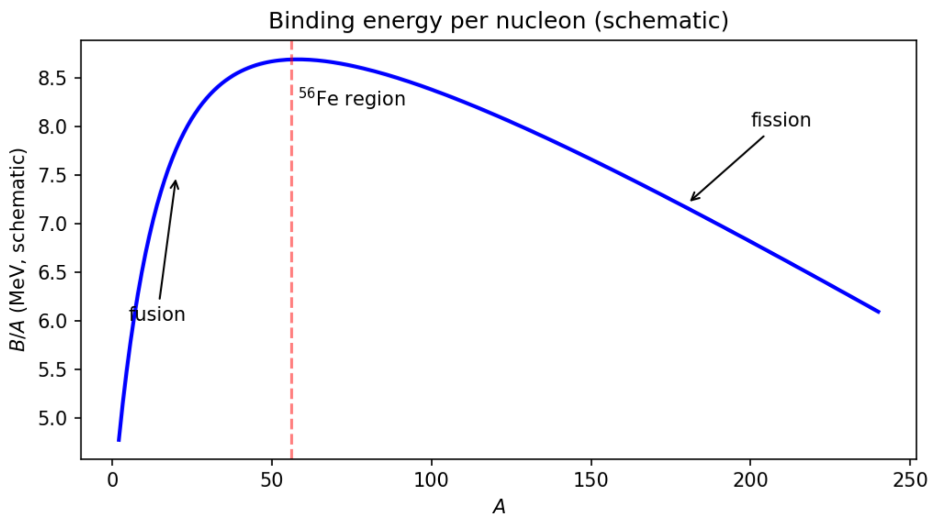

Figure 1.1: Binding energy per nucleon B/A versus mass number A. The peak near iron/nickel marks the most tightly bound nuclei. Fusion of light nuclei (left of the peak) and fission of heavy nuclei (right of the peak) both move systems toward higher B/A and release energy. Re-extracted from PH5001 lecture material (Prof. Bharat Kumar).

Figure 1.1 summarises nuclear energetics:

- B/A rises steeply from the lightest nuclei toward $A \sim 56$–62 (Fe, Ni).
- After the peak, B/A decreases slowly through the heavy elements.

Any nuclear rearrangement that increases the total binding energy decreases the rest mass and releases energy $Q = (\Delta M)c^2$.

Fission (right of the peak). A heavy parent ($A \sim 240$, $B/A \sim 7.6\,\mathrm{MeV}$) splits into medium-mass fragments nearer the peak ($B/A \sim 8.5\,\mathrm{MeV}$). Energy is released; that was Chapters **??**–**??**.

Fusion (left of the peak). Two light nuclei combine into a heavier product still left of or nearer the peak. The product has larger B/A than the reactants, so energy is released. The classic laboratory example is

$$\mathrm{D} + \mathrm{T} \;\rightarrow\; {}^4\mathrm{He} + n + 17.6\,\mathrm{MeV}, \tag{1.1}$$

and in stars the *pp* chain begins with $p + p \rightarrow \mathrm{d} + e^+ + \nu_e$.

Key idea

Fusion: light nuclei join and move up the B/A curve toward iron. Fission: heavy nuclei split and move up the B/A curve from the other side. Both release energy; both are blocked by barriers (Coulomb for fusion, deformation for fission).

Beyond iron, fusion would *decrease* B/A and consume energy; those endothermic fusions occur in supernovae only because of the enormous free energy of the explosion, not as a steady power source.

1.2 Fusion Q-value from binding energies

For a general fusion reaction

$$X + Y \rightarrow Z + \cdots, \tag{1.2}$$

the Q-value is

$$Q = [M_X + M_Y - M_Z - \cdots]c^2. \tag{1.3}$$

Writing each mass as $Mc^2 = \sum m_N c^2 - E_B$ and cancelling free nucleon masses (when nucleon number is conserved) yields the binding form

$$Q = E_B(\text{final}) - E_B(\text{initial}). \tag{1.4}$$

Fusion releases energy when the product is more tightly bound than the sum of the reactants: $E_B(\text{final}) > E_B(\text{initial})$.

1.2.1 Worked structure: ${}^{20}\mathrm{Ne} + {}^{20}\mathrm{Ne} \rightarrow {}^{40}\mathrm{Ca}$

Consider

$${}^{20}_{10}\mathrm{Ne} + {}^{20}_{10}\mathrm{Ne} \;\rightarrow\; {}^{40}_{20}\mathrm{Ca} + \gamma. \tag{1.5}$$

(The γ carries most of the energy if the calcium is formed in a bound state that radiatively de-excites; the mass bookkeeping is the same.)

Rest energy before fusion. One neon nucleus has rest energy

$$E(^{20}\mathrm{Ne}) = (10m_p + 10m_n)c^2 - E_B(^{20}\mathrm{Ne}). \tag{1.6}$$

Two neon nuclei contribute

$$E_{\text{before}} = 2(10m_p + 10m_n)c^2 - 2E_B(^{20}\mathrm{Ne}). \tag{1.7}$$

Rest energy after fusion.

$$E(^{40}\mathrm{Ca}) = (20m_p + 20m_n)c^2 - E_B(^{40}\mathrm{Ca}). \tag{1.8}$$

Q-value.

$$\begin{aligned} Q &= E_{\text{before}} - E_{\text{after}} \\ &= 2(10m_p + 10m_n)c^2 - 2E_B(\mathrm{Ne}) - \left[(20m_p + 20m_n)c^2 - E_B(\mathrm{Ca})\right] \\ &= E_B(^{40}\mathrm{Ca}) - 2E_B(^{20}\mathrm{Ne}). \end{aligned} \tag{1.9}$$

Equivalently, in terms of binding per nucleon,

$$Q = 40\left[\left(\frac{B}{A}\right)_{\mathrm{Ca}} - \left(\frac{B}{A}\right)_{\mathrm{Ne}}\right]. \tag{1.10}$$

Since $(B/A)_{\mathrm{Ca}} > (B/A)_{\mathrm{Ne}}$ on the rising part of the curve, $Q > 0$. Numerically the difference is of order

$$Q \sim 20\text{–}25\,\mathrm{MeV} \tag{1.11}$$

for this reaction (lecture estimate $\sim 23\,\mathrm{MeV}$). The released energy appears as excitation of ^{40}Ca, photon emission, and recoil [6, 1, 14].

Remark

Equation (1.4) is the same identity used for fission ($Q = E_{B,\text{frag}} - E_{B,\text{parent}}$). Only the direction of motion on the B/A curve changes.

1.3 The Coulomb barrier

Although fusion of light nuclei is energetically favoured, the nuclei are positively charged. They repel until they approach within the range of the attractive nuclear force ($\sim$ a few fm). The electrostatic potential energy at centre-to-centre separation r is

$$V_{\mathrm{C}}(r) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0\, r}. \tag{1.12}$$

As r decreases, V_C rises. When the nuclear surfaces touch, the nuclear attraction rapidly dominates and the potential drops into a deep well. The maximum of the potential (or the value of V_C at contact in a sharp-surface model) is the **Coulomb barrier height**

$$\boxed{V_{\max} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 (r_1 + r_2)}.} \tag{1.13}$$

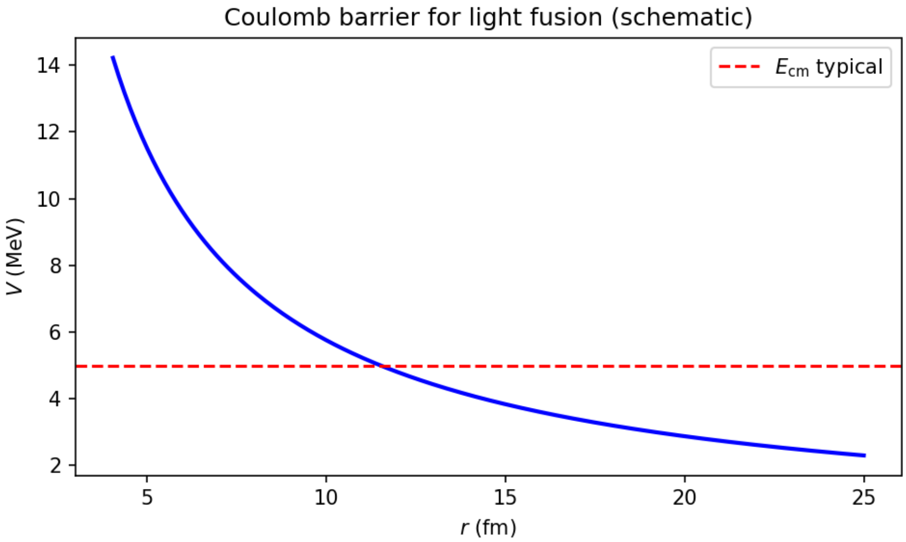

Figure 1.2: Coulomb barrier for fusion: potential energy versus separation r. Outside contact the potential is $\propto 1/r$; inside, the nuclear force creates a deep attractive well. Fusion at $E \ll V_{\max}$ proceeds by tunneling. Re-extracted from PH5001 lecture material.

Nuclear radii follow the empirical law

$$r = r_0 A^{1/3}, \qquad r_0 \approx 1.2\,\text{fm}. \tag{1.14}$$

A convenient numerical constant is

$$\frac{e^2}{4\pi\varepsilon_0} \approx 1.44\,\text{MeV}\cdot\text{fm}, \tag{1.15}$$

so

$$V_{\max} \approx \frac{Z_1 Z_2 \times 1.44\,\text{MeV}\cdot\text{fm}}{r_1 + r_2}. \tag{1.16}$$

1.3.1 Example: neon–neon

For two ^{20}Ne nuclei, $Z_1 = Z_2 = 10$, $A = 20$:

$$r_1 = r_2 = 1.2 \times 20^{1/3}\,\text{fm} \approx 1.2 \times 2.71\,\text{fm} \approx 3.26\text{–}3.5\,\text{fm} \tag{1.17}$$

(using the lecture's 3.5 fm estimate). Contact separation $r_1 + r_2 \approx 7.0\,\text{fm}$, so

$$V_{\max} = \frac{10 \times 10 \times 1.44\,\text{MeV}\cdot\text{fm}}{7.0\,\text{fm}} \approx 20.6\,\text{MeV} \sim 20\,\text{MeV}. \tag{1.18}$$

Classically one needs $\sim 20\,\text{MeV}$ of relative kinetic energy to surmount the barrier.

1.3.2 Example: proton–proton

For $p + p$, $Z_1 = Z_2 = 1$. Taking a contact separation of order $r_1 + r_2 \approx 2\,\text{fm}$ (two proton radii),

$$V_{\text{max}} = \frac{1 \times 1 \times 1.44\,\text{MeV} \cdot \text{fm}}{2\,\text{fm}} = 0.72\,\text{MeV}. \tag{1.19}$$

The pp barrier is much lower, which is why hydrogen fusion is the first chain to ignite in stars. Still, $0.72\,\text{MeV}$ is huge compared with thermal energies in the solar core (Section 1.4) [6, 1, 14, 2].

Key idea

$V_{\text{max}} = Z_1 Z_2 e^2/[4\pi\varepsilon_0(r_1 + r_2)]$, $r = r_0 A^{1/3}$. Ne + Ne: $V_{\text{max}} \sim 20\,\text{MeV}$. $p + p$: $V_{\text{max}} \sim 0.72\,\text{MeV}$. Barrier height grows rapidly with $Z_1 Z_2$.

1.4 Classical threshold vs quantum tunneling

1.4.1 Classical requirement

Classically, fusion requires relative kinetic energy

$$E_k > V_{\text{max}} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0(r_1 + r_2)}. \tag{1.20}$$

Accelerators can supply tens of MeV and drive fusion reactions for nuclear physics studies, but the energy invested in the beams typically exceeds the energy recovered from Q: accelerators are not practical power plants.

1.4.2 Quantum tunneling

In quantum mechanics a particle with energy $E < V_{\text{max}}$ has a nonzero probability to penetrate the barrier. The tunneling probability is of Gamow form

$$P \sim e^{-2\gamma}, \tag{1.21}$$

where the Gamow exponent γ (sometimes written G) is an integral of the local wave number under the barrier. For a pure Coulomb barrier a standard result is

$$\gamma \propto \frac{Z_1 Z_2 e^2}{\hbar v} \qquad (\text{schematically } \gamma \sim Z_1 Z_2 e^2/(\hbar v)), \tag{1.22}$$

with v the relative speed at infinity. As v increases, γ falls and P rises exponentially. This is the same physics as α decay (Lectures 16–17), run in reverse: instead of tunneling *out*, the nuclei tunnel *in* [6, 14].

1.4.3 Thermonuclear fusion

In a hot plasma the kinetic energy is thermal. The mean kinetic energy of a particle is

$$\langle E_k \rangle = \frac{3}{2} k_B T. \tag{1.23}$$

Fusion powered by the thermal motion of a plasma is called **thermonuclear fusion**. It is the energy source of main-sequence stars and the goal of magnetic and inertial confinement fusion devices.

Room temperature. At $T \approx 300\,\mathrm{K}$,

$$k_B T \approx 0.026\,\mathrm{eV}, \tag{1.24}$$

utterly negligible compared with even the *pp* barrier of 720 keV. The tunneling probability is zero for practical purposes.

Temperature for $E_k \sim 150\,\mathrm{keV}$. Set $\frac{3}{2} k_B T = 150\,\mathrm{keV}$, so $k_B T = 100\,\mathrm{keV}$:

$$T = \frac{100 \times 10^3\,\mathrm{eV}}{8.617 \times 10^{-5}\,\mathrm{eV/K}} \approx 1.16 \times 10^9\,\mathrm{K}. \tag{1.25}$$

Temperatures of order 10^8–10^9 K are the thermonuclear regime for laboratory fusion concepts.

Solar core. The Sun's core has $T \approx 1.5 \times 10^7\,\mathrm{K}$, so

$$k_B T \approx 1.3\,\mathrm{keV}, \qquad \langle E_k \rangle \sim 2\,\mathrm{keV} \tag{1.26}$$

(order 1 keV in the lecture's estimate). This is ~ 700 times smaller than $V_{\max}(pp) \approx 720\,\mathrm{keV}$. Fusion in the Sun is possible only because of tunneling, aided by two further facts:

(i) the Maxwell–Boltzmann distribution has a high-energy tail, so some protons have $E \gg \langle E_k \rangle$;

(ii) the solar core is extremely dense ($\rho \sim 150\,\mathrm{g\,cm^{-3}}$) and huge, so even a tiny rate per pair yields enormous luminosity.

Caution

Do not equate $k_B T$ with the barrier height. Stars and fusion reactors operate with $k_B T \ll V_{\max}$; the Gamow tunneling factor and the Maxwell tail jointly set the rate (Chapter **??**).

1.5 Maxwell tail and barrier penetration

In a gas at temperature T the speed distribution is Maxwellian. A fraction of particles has kinetic energy many times k_BT:

$$E \sim 10\,k_BT,\ 20\,k_BT,\ \ldots \tag{1.27}$$

Those particles tunnel far more readily because $\gamma \propto 1/v \propto 1/\sqrt{E}$. The fusion rate is therefore dominated by a compromise between the falling Maxwell factor e^{-E/k_BT} and the rising penetration factor $e^{-b/\sqrt{E}}$. The product peaks at the **Gamow peak**, derived properly in the next chapter.

The barrier penetration probability for energy E may be written as a WKB integral between classical turning points r_1 and r_2:

$$P \sim \exp\left(-2\int_{r_1}^{r_2}\sqrt{\frac{2m}{\hbar^2}(V(r)-E)}\,\mathrm{d}r\right), \tag{1.28}$$

with $V(r) = Z_1Z_2e^2/(4\pi\varepsilon_0 r)$ outside the nuclear well. Increasing E shrinks the integration range and the integrand, so P grows rapidly with energy [6, 14, 2].

1.6 Reaction rate preview

The number of fusion reactions per unit volume per unit time between species X and Y is

$$R = n_X\,n_Y\,\langle\sigma v\rangle, \tag{1.29}$$

where n_X, n_Y are number densities, σ is the fusion cross section, v is the relative velocity, and $\langle\sigma v\rangle$ is the thermal average over the Maxwell distribution. The energy generation rate is $R \times Q$ per unit volume. Concentrations and temperature therefore control stellar luminosity and the performance of fusion devices. The full derivation of R from beam–target geometry, the $1/v^2$ low-energy cross section, and the Gamow peak occupies Chapter **??**.

Takeaway

- B/A peaks at Fe/Ni: fusion left of peak, fission right of peak.
- $Q = E_B(\text{final}) - E_B(\text{initial})$; for Ne + Ne → Ca, $Q = E_B(\text{Ca}) - 2E_B(\text{Ne}) \sim 23\,\text{MeV}$.
- Coulomb barrier $V_{\max} = Z_1Z_2e^2/[4\pi\varepsilon_0(r_1+r_2)]$; Ne + Ne $\sim 20\,\text{MeV}$, $pp \sim 0.72\,\text{MeV}$.
- Thermonuclear fusion: $E \sim k_BT \ll V_{\max}$; tunneling $P \sim e^{-2\gamma}$; solar core $T \sim 1.5\times10^7\,\text{K}$, $E \sim 1\,\text{keV}$.
- Rate $R = n_Xn_Y\langle\sigma v\rangle$ (derived next).

1.7 Further physics from the literature

Stellar burning stages. Hydrogen burning (*pp* chain and CNO cycle) dominates main-sequence stars. When core hydrogen is exhausted, contraction raises T until helium burning ($3\alpha \to {}^{12}\mathrm{C}$) ignites, then carbon, neon, oxygen, and silicon burning in massive stars, ending at the iron peak where fusion no longer releases energy [14, 6, 2].

Screening and plasma effects. In a dense plasma the Coulomb potential is screened by surrounding electrons and ions, slightly reducing the effective barrier and enhancing the rate relative to the bare Gamow factor. Screening corrections matter in precision solar models and in white-dwarf interiors [14].

Laboratory fusion approaches. Magnetic confinement (tokamaks, stellarators) and inertial confinement (laser or ion-beam driven pellets) both aim for thermonuclear $\mathrm{D}+\mathrm{T}$ conditions. Neither is required to reach $E = V_{\max}$; both rely on tunneling at $T \sim 10^8\,\mathrm{K}$ [1, 2].

Connection to α decay. The Gamow factor for fusion is the time-reversed cousin of the Gamow factor for α emission (Chapters **??**–**??** if present in the volume). The same WKB integral appears with incoming versus outgoing boundary conditions [6, 14].

Reading guide

Krane [6] (fusion overview and barriers); Dunlap [1] (energetics); Basdevant [14] (stellar fusion and S-factors); Demtröder [2] (experimental and astrophysical context).

1.8 Problems with solutions

Problem 25.1 — Q from B/A for $\mathrm{D}+\mathrm{T}$

The reaction $\mathrm{D}+\mathrm{T} \to {}^4\mathrm{He} + n$ has $Q = 17.6\,\mathrm{MeV}$. If the total binding energy of $\mathrm{D}+\mathrm{T}$ is B_i and that of ${}^4\mathrm{He}$ is $B_\alpha = 28.3\,\mathrm{MeV}$ (the neutron is unbound as a free particle, $B_n = 0$), find B_i from $Q = B_f - B_i$ and comment [6, 1].

Solution 25.1

Step 1. Final binding: $B_f = B_\alpha + B_n = 28.3\,\mathrm{MeV}$.
Step 2.

$$Q = B_f - B_i \quad \Rightarrow \quad B_i = B_f - Q = 28.3 - 17.6 = 10.7\,\mathrm{MeV}.$$

That matches $B(\mathrm{d}) + B(\mathrm{t}) \approx 2.2 + 8.5 = 10.7\,\mathrm{MeV}$. Fusion increases total binding by $17.6\,\mathrm{MeV}$.

Problem 25.2 — Neon fusion Q structure

Starting from rest energies expressed with binding energies, derive $Q = E_B({}^{40}\mathrm{Ca}) - 2E_B({}^{20}\mathrm{Ne})$ for ${}^{20}\mathrm{Ne} + {}^{20}\mathrm{Ne} \to {}^{40}\mathrm{Ca}$. If $(B/A)_{\mathrm{Ca}} - (B/A)_{\mathrm{Ne}} = 0.58\,\mathrm{MeV}$, compute Q.

Solution 25.2

Step 1. Follow Eqs. (1.7)–(1.9): nucleon mass terms cancel, leaving $Q = E_B(\mathrm{Ca}) - 2E_B(\mathrm{Ne})$.
Step 2. From Eq. (1.10),

$$Q = 40 \times 0.58\,\mathrm{MeV} = 23.2\,\mathrm{MeV},$$

consistent with the lecture estimate $\sim 23\,\mathrm{MeV}$ [14, 1].

Problem 25.3 — Coulomb barrier for $^{12}\mathrm{C} + {}^{12}\mathrm{C}$

Estimate $V_{\max}$ for two ^{12}C nuclei. Use $r = 1.2\,A^{1/3}\,\mathrm{fm}$ and $e^2/(4\pi\varepsilon_0) = 1.44\,\mathrm{MeV}\cdot\mathrm{fm}$.

Solution 25.3

Step 1.

$$r = 1.2 \times 12^{1/3} = 1.2 \times 2.29 \approx 2.75\,\mathrm{fm}, \qquad r_1 + r_2 = 5.50\,\mathrm{fm}.$$

Step 2.

$$V_{\max} = \frac{6 \times 6 \times 1.44}{5.50} = \frac{51.84}{5.50} \approx 9.4\,\mathrm{MeV}.$$

Carbon burning in massive stars must tunnel a $\sim 10\,\mathrm{MeV}$ barrier [6, 14].

Problem 25.4 — pp barrier and solar k_BT

Show that $V_{\max}(pp) \approx 0.72\,\mathrm{MeV}$ for $r_1 + r_2 = 2\,\mathrm{fm}$. At $T = 1.5 \times 10^7\,\mathrm{K}$, compute k_BT in keV and the ratio $V_{\max}/k_BT$.

Solution 25.4

Step 1.

$$V_{\max} = \frac{1.44\,\mathrm{MeV}\cdot\mathrm{fm}}{2\,\mathrm{fm}} = 0.72\,\mathrm{MeV}.$$

Step 2.

$$k_BT = (8.617 \times 10^{-5}\,\mathrm{eV/K}) \times 1.5 \times 10^7\,\mathrm{K} \approx 1.29 \times 10^3\,\mathrm{eV} \approx 1.3\,\mathrm{keV}.$$

Step 3.

$$\frac{V_{\max}}{k_BT} = \frac{720\,\mathrm{keV}}{1.3\,\mathrm{keV}} \approx 550.$$

Classical surmounting is impossible; solar fusion is pure tunneling physics [6, 2].

Problem 25.5 — Temperature for 150 keV

What temperature T gives $\frac{3}{2}k_BT = 150\,\mathrm{keV}$? Compare with the solar core temperature.

Solution 25.5

$$k_BT = 100\,\mathrm{keV}, \qquad T = \frac{100\times 10^3}{8.617\times 10^{-5}} \approx 1.16\times 10^9\,\mathrm{K}.$$

This is about 80 times hotter than the solar core ($\sim 1.5\times 10^7\,\mathrm{K}$). Laboratory fusion concepts operate near 10^8 K with tunneling still essential [1, 14].

Problem 25.6 — Fusion vs fission energy per nucleon

Compare Q/A for (a) fission with $Q = 200\,\mathrm{MeV}$, $A = 236$; (b) D + T fusion with $Q = 17.6\,\mathrm{MeV}$ and 5 nucleons in the reactants. Which releases more energy per nucleon?

Solution 25.6

(a)

$$\frac{Q}{A} = \frac{200}{236} \approx 0.85\,\mathrm{MeV/nucleon}.$$

(b)

$$\frac{Q}{A} = \frac{17.6}{5} = 3.52\,\mathrm{MeV/nucleon}.$$

Fusion of light nuclei releases several times more energy *per nucleon* than fission, which is why fusion fuels are attractive despite the barrier challenge [6, 1].

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