Chapter 1

Nuclear Reaction Kinematics and Spectroscopy

> *"The energy of the outgoing particle is a spectrometer for the excitation of the residual nucleus, once Q-value kinematics are under control."*
>
> reaction spectroscopy [6, 1]

Chapter **??** classified reaction mechanisms. This chapter connects measured kinetic energies and angles to nuclear excitation energies: two-body momentum and energy conservation, the Q-value, the full non-relativistic relation among K_a, K_b, θ, and masses, thresholds for endothermic reactions, neutron capture on ^{238}U as a worked example, and the idea of resonant energy dependence of the cross section [6, 1, 14, 2].

1.1 Experimental geometry and what is measured

Consider the binary reaction

$$X(a,b)Y \qquad \text{or} \qquad a + X \to b + Y \tag{1.1}$$

in the laboratory frame with target X at rest. Projectile a has kinetic energy K_a and momentum $\mathbf{p}_a$. Product b is detected at laboratory angle θ with kinetic energy K_b and momentum $\mathbf{p}_b$. Residual Y recoils at angle ϕ with kinetic energy K_Y and momentum $\mathbf{p}_Y$.

angular distribution of the particle. The detector also measures the kinetic energy of the b-particles, allowing us to analyze the distribution of their energy as they reach the detector. This provides the experimental data.

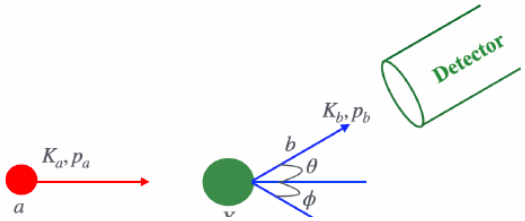

Figure 1.1: Laboratory setup for a two-body reaction $X(a,b)Y$: projectile a, target X, light product b at angle θ, residual Y at angle ϕ, and a detector that measures K_b (lecture figure).

Measurable quantities are typically K_a (set by the accelerator), θ, and K_b (from the detector). The residual kinetic energy K_Y is usually not measured directly. Nuclear data tables supply rest masses. The unknown of spectroscopic interest is often the excitation energy ΔE of Y [6, 1].

1.2 Conservation laws

1.2.1 Momentum

With the beam along $+x$ and the reaction in the xy plane,

$$p_a = p_b \cos\theta + p_Y \cos\phi, \tag{1.2}$$

$$0 = p_b \sin\theta - p_Y \sin\phi. \tag{1.3}$$

(The target is at rest, so initial $p_y = 0$.)

1.2.2 Energy and Q-value

Including rest energies and allowing residual excitation ΔE,

$$m_a c^2 + m_X c^2 + K_a = m_b c^2 + m_Y c^2 + K_b + K_Y + \Delta E. \tag{1.4}$$

Define the reaction **Q-value** from ground-state rest masses,

$$Q \equiv (m_a + m_X - m_b - m_Y)c^2. \tag{1.5}$$

Then (1.4) becomes

$$Q + K_a = K_b + K_Y + \Delta E. \tag{1.6}$$

- $Q > 0$: **exoergic** (exothermic); energy is released.
- $Q < 0$: **endoergic** (endothermic); a minimum beam energy is required (Section 1.5).

If Y is left in its ground state, set $\Delta E = 0$. Excited states appear as discrete peaks at lower K_b for fixed θ [6, 1, 14].

Physical meaning of ΔE. Each bound level of Y produces a kinematic locus $K_b(\theta)$. Measuring K_b at known θ and K_a is nuclear level spectroscopy via reaction kinematics.

1.3 Eliminating the unobserved residual

Nuclear kinetic energies of a few MeV on mass $\sim A \times 931$ MeV justify the non-relativistic relations

$$K_Y = \frac{p_Y^2}{2m_Y}, \qquad p_a^2 = 2m_a K_a, \qquad p_b^2 = 2m_b K_b. \tag{1.7}$$

From (1.2)–(1.3),

$$p_Y \cos\phi = p_a - p_b \cos\theta, \qquad p_Y \sin\phi = p_b \sin\theta. \tag{1.8}$$

Squaring and adding eliminates ϕ:

$$p_Y^2 = (p_a - p_b\cos\theta)^2 + (p_b\sin\theta)^2 = p_a^2 + p_b^2 - 2p_a p_b\cos\theta. \tag{1.9}$$

Therefore

$$K_Y = \frac{p_Y^2}{2m_Y} = \frac{m_a}{m_Y}K_a + \frac{m_b}{m_Y}K_b - \frac{2}{m_Y}\sqrt{m_a m_b K_a K_b}\,\cos\theta. \tag{1.10}$$

Substitute (1.10) into (1.6):

$$Q + K_a = K_b + \frac{m_a}{m_Y}K_a + \frac{m_b}{m_Y}K_b - \frac{2}{m_Y}\sqrt{m_a m_b K_a K_b}\,\cos\theta + \Delta E. \tag{1.11}$$

Collecting terms,

$$Q + K_a\Big(1 - \frac{m_a}{m_Y}\Big) = K_b\Big(1 + \frac{m_b}{m_Y}\Big) - \frac{2}{m_Y}\sqrt{m_a m_b K_a K_b}\,\cos\theta + \Delta E. \tag{1.12}$$

Solving for the excitation energy,

$$\Delta E = Q - K_b\Big(1 + \frac{m_b}{m_Y}\Big) + K_a\Big(1 - \frac{m_a}{m_Y}\Big) + \frac{2}{m_Y}\sqrt{m_a m_b K_a K_b}\,\cos\theta. \tag{1.13}$$

All quantities on the right are either tabulated (Q, masses) or measured (K_a, K_b, θ). Equation (1.13) is the working formula of two-body reaction spectroscopy in the non-relativistic limit [6, 1].

Caution

Masses in (1.13) must be consistent (nuclear or atomic with the same convention on both sides of Q). For precise work one uses relativistic kinematics; the structure of the argument is unchanged [6, 14].

1.4 Quadratic equation for K_b and kinematic loci

For fixed K_a, θ, and ΔE, equation (1.11) is quadratic in $\sqrt{K_b}$ (or in K_b after clearing the square root). Writing $x = \sqrt{K_b}$,

$$\Big(1 + \frac{m_b}{m_Y}\Big)x^2 - \frac{2\cos\theta}{m_Y}\sqrt{m_a m_b K_a}\;x + \Big[K_a\Big(\frac{m_a}{m_Y} - 1\Big) - Q + \Delta E\Big] = 0. \tag{1.14}$$

Real positive K_b requires a non-negative discriminant. That condition restricts allowed $(K_a, \theta, \Delta E)$ and underlies threshold formulae [6, 1].

1.5 Threshold for endothermic reactions

If $Q < 0$ and $\Delta E \geq 0$, energy conservation in the CM frame demands

$$E_{\mathrm{cm}} \geq -Q + \Delta E. \tag{1.15}$$

With $E_{\rm cm} = m_X K_a/(m_a + m_X)$ for a stationary target,

$$K_a^{\rm th} = (-Q + \Delta E)\frac{m_a + m_X}{m_X} = (-Q + \Delta E)\Big(1 + \frac{m_a}{m_X}\Big). \tag{1.16}$$

The factor $(1 + m_a/m_X) > 1$ appears because part of the beam energy is tied up in CM motion and is unavailable for creating rest mass or excitation. Even for $Q > 0$, producing an excited residual with $\Delta E > |Q|$ can create an effective threshold [6, 1, 14].

Exoergic case. If $Q - \Delta E > 0$, the reaction is allowed down to $K_a \to 0$ (neglecting Coulomb barrier and angular-momentum constraints). Charged projectiles may still need barrier penetration energy even when $Q > 0$.

1.6 Example: neutron capture on ^{238}U

Consider radiative capture

$$n + {}^{238}\mathrm{U} \longrightarrow {}^{239}\mathrm{U}^* \longrightarrow {}^{239}\mathrm{U} + \gamma. \tag{1.17}$$

Using tabulated rest masses, the ground-state Q-value is

$$Q = \big(m_n + m(^{238}\mathrm{U}) - m(^{239}\mathrm{U})\big)c^2 \approx 4.86\,\mathrm{MeV}. \tag{1.18}$$

In the laboratory the target is at rest. Momentum conservation requires the compound nucleus to carry the neutron's momentum p_n. The residual kinetic energy is

$$K_U = \frac{p_n^2}{2m_U} = K_n\frac{m_n}{m_U}. \tag{1.19}$$

Energy balance for forming ^{239}U at excitation ΔE reads

$$K_n + Q = K_U + \Delta E = K_n\frac{m_n}{m_U} + \Delta E, \tag{1.20}$$

so

$$\Delta E = Q + K_n\Big(1 - \frac{m_n}{m_U}\Big) \approx Q + K_n \qquad (m_n \ll m_U). \tag{1.21}$$

For a **thermal** neutron ($K_n \approx 0$),

$$\Delta E \approx Q \approx 4.86\,\mathrm{MeV}. \tag{1.22}$$

Capture therefore populates levels of ^{239}U near 4.86 MeV of excitation. Discrete levels above that energy are reached by increasing K_n. When K_n matches a compound-nucleus resonance, the capture cross section rises sharply. The energy dependence of $\sigma(K_n)$ shows a series of peaks: **neutron resonances**. Levels below ~ 4.86 MeV are not populated by capture of free neutrons, because even $K_n = 0$ already deposits $\sim Q$ [6, 1, 2].

Remark

The same $Q \approx 4.86\,\mathrm{MeV}$ sets the neutron separation energy of ^{239}U: $S_n(^{239}\mathrm{U}) = Q$. Resonances just above S_n dominate low-energy neutron physics of ^{238}U and enter reactor design [6, 1].

1.7 Cross section versus energy: resonances (concept)

Plotting σ against incident energy for a fixed channel often reveals narrow peaks. Each peak corresponds to a quasi-stationary state of the compound system whose energy matches the available CM energy. Qualitatively:

- on resonance, the reaction probability is large;
- off resonance, other processes (potential scattering, direct channels) may dominate.

The line shape is described by the **Breit–Wigner** formula (derived from a resonant propagator / R-matrix theory). For an isolated resonance,

$$\sigma(E) \propto \frac{\Gamma_i \Gamma_f}{(E - E_R)^2 + (\Gamma/2)^2}, \tag{1.23}$$

where E_R is the resonance energy, Γ_i and Γ_f are partial widths for entrance and exit channels, and $\Gamma = \sum \Gamma_c$ is the total width. The lifetime of the compound state is

$$\tau \sim \frac{\hbar}{\Gamma} : \tag{1.24}$$

narrow resonances live longer. Full derivation of (1.23) is left to a later treatment; the kinematic message of this chapter is that discrete ΔE values map to discrete K_b peaks and to discrete incident energies for formation resonances [6, 14, 2].

Key idea

Students should remember:

- $Q = (m_a + m_X - m_b - m_Y)c^2$; $Q + K_a = K_b + K_Y + \Delta E$.
- Eliminate K_Y with $p_Y^2 = p_a^2 + p_b^2 - 2p_a p_b \cos\theta$.
- ΔE from measured K_b, θ, K_a via (1.13).
- Endothermic threshold: $K_a^{\mathrm{th}} = (-Q + \Delta E)(1 + m_a/m_X)$.
- $n + {}^{238}$U: $Q \sim 4.86\,\mathrm{MeV}$ sets capture excitation for slow neutrons; resonances map levels above S_n.
- Cross section peaks at compound resonances (Breit–Wigner shape).

1.8 Further physics from the literature

Relativistic two-body kinematics. At beam energies of tens to hundreds of MeV per nucleon the non-relativistic reduction $p^2 = 2mK$ fails. Exact Lorentz-invariant s-channel kinematics replace (1.13); modern analysis codes implement the full transformation between laboratory and CM frames [6, 14].

Missing-mass spectroscopy. When only particle b is detected, the missing mass of Y is reconstructed from four-momentum conservation. Peaks in the missing-mass spectrum are excited states of Y. The method is standard at intermediate-energy facilities [6, 1].

R-matrix and Breit–Wigner parameters. Isolated resonances are parameterised by E_R, partial widths Γ_c, and channel radii in R-matrix theory. Overlapping resonances and direct–resonance interference require more elaborate treatments [6, 14, 2].

Reactor and astrophysical capture. The $^{238}\mathrm{U}(n,\gamma)$ resonances control neutron economy in reactors. In astrophysics, (n,γ) rates on unstable nuclei set the s- and r-process paths; many rates are still extrapolated from statistical models constrained by stable-target data [6, 1, 2].

Reading guide

Krane [6] Ch. 11 (kinematics, Q-values, thresholds, resonances); Dunlap [1] (worked kinematic examples); Basdevant [14] (resonances); Demtröder [2] (neutron physics). Mass tables: AME / NuDat [20, 21].

1.9 Problems with solutions

Problem 22.1: Q-value from masses

Using rest energies $m_a c^2 = 938.3\,\mathrm{MeV}$, $m_X c^2 = 11174.9\,\mathrm{MeV}$, $m_b c^2 = 1875.6\,\mathrm{MeV}$, $m_Y c^2 = 10252.5\,\mathrm{MeV}$, compute Q for $a + X \to b + Y$. Is the reaction exoergic or endoergic? [6]

Solution 22.1

$$Q = (938.3 + 11174.9) - (1875.6 + 10252.5) = 12113.2 - 12128.1 = -14.9\,\mathrm{MeV}.$$

Endoergic ($Q < 0$).

Problem 22.2: Threshold energy

For a reaction with $Q = -14.9\,\mathrm{MeV}$, $\Delta E = 0$, projectile mass number $A_a = 1$, target $A_X = 12$, find the laboratory threshold K_a^{th} [6, 1].

Solution 22.2

$$K_a^{\mathrm{th}} = (-Q)\Big(1 + \frac{m_a}{m_X}\Big) = 14.9\Big(1 + \frac{1}{12}\Big) = 14.9 \times \frac{13}{12} = 16.1\,\mathrm{MeV}.$$

Problem 22.3: Residual momentum

In a reaction at $\theta = 30^\circ$, $p_a = 150\,\mathrm{MeV}/c$, $p_b = 120\,\mathrm{MeV}/c$. Find p_Y from momentum conservation [1].

Solution 22.3

$$p_Y^2 = p_a^2 + p_b^2 - 2p_a p_b \cos\theta = 150^2 + 120^2 - 2(150)(120)\cos 30^\circ.$$

$$\cos 30^\circ = \sqrt{3}/2 \approx 0.866, \qquad 2(150)(120)(0.866) = 31176.$$

$$p_Y^2 = 22500 + 14400 - 31176 = 5724, \qquad p_Y = 75.7\,\mathrm{MeV}/c.$$

Problem 22.4: Excitation from K_b (forward angle)

For $n + {}^{12}\mathrm{C} \to n' + {}^{12}\mathrm{C}^*$ treat the inelastic neutron as $a = b = n$, $X = Y = {}^{12}\mathrm{C}$, $Q = 0$. At $\theta = 0$ and $K_a = 20.0\,\mathrm{MeV}$, a peak is found at $K_b = 15.5\,\mathrm{MeV}$. Estimate ΔE using $\Delta E \approx K_a - K_b$ (heavy residual, forward kinematics), then correct with the leading recoil terms of (1.13) using $m_n/m_C = 1/12$ [6].

Solution 22.4

Crude estimate:

$$\Delta E \approx 20.0 - 15.5 = 4.5\,\mathrm{MeV}.$$

With $Q = 0$, $m_a = m_b = m_n$, $m_Y = m_C$, $\theta = 0$, $\cos\theta = 1$,

$$\begin{aligned}\Delta E &= -K_b\Big(1 + \frac{1}{12}\Big) + K_a\Big(1 - \frac{1}{12}\Big) + \frac{2}{12}\sqrt{K_a K_b} \\ &= -15.5 \times \frac{13}{12} + 20.0 \times \frac{11}{12} + \frac{1}{6}\sqrt{20 \times 15.5}.\end{aligned}$$

$$\sqrt{310} \approx 17.61, \quad \frac{1}{6} \times 17.61 \approx 2.94,$$

$$\Delta E = -16.79 + 18.33 + 2.94 = 4.48\,\mathrm{MeV}.$$

The residual is excited by about $4.5\,\mathrm{MeV}$.

Problem 22.5: Thermal capture on ^{238}U

Show that a neutron with $K_n = 0.025\,\mathrm{eV}$ on $^{238}\mathrm{U}$ produces essentially the same excitation as a zero-energy neutron if $Q = 4.86\,\mathrm{MeV}$. Compute ΔE from (1.21) [6, 1].

Solution 22.5

$$\frac{m_n}{m_U} \approx \frac{1}{239} \approx 4.2 \times 10^{-3}, \qquad K_n = 2.5 \times 10^{-8}\,\mathrm{MeV}.$$

$$\Delta E = Q + K_n\Big(1 - \frac{m_n}{m_U}\Big) = 4.86\,\mathrm{MeV} + 2.5 \times 10^{-8}\,\mathrm{MeV} \times 0.996 \approx 4.86\,\mathrm{MeV}.$$

Thermal kinetic energy is negligible next to Q.

Problem 22.6: Resonance and lifetime

A neutron resonance has total width $\Gamma = 0.10\,\text{eV}$. Estimate the compound-state lifetime $\tau = \hbar/\Gamma$. Use $\hbar = 6.58 \times 10^{-16}\,\text{eV}\cdot\text{s}$ [6, 14].

Solution 22.6

$$\tau = \frac{6.58 \times 10^{-16}\,\text{eV}\cdot\text{s}}{0.10\,\text{eV}} = 6.6 \times 10^{-15}\,\text{s}.$$

This lies in the compound-nucleus range 10^{-16}–$10^{-15}\,\text{s}$, far longer than a direct transit time $\sim 10^{-22}\,\text{s}$.

Problem 22.7: Why excited residuals lower K_b

From $Q + K_a = K_b + K_Y + \Delta E$, explain why peaks corresponding to higher ΔE appear at lower K_b in a fixed-angle spectrum for fixed K_a [6, 1].

Solution 22.7

For fixed entrance energy K_a and fixed Q, the sum $K_b + K_Y + \Delta E$ is fixed. Increasing ΔE reduces the kinetic energy available to share between b and Y. At a fixed angle the kinematic solution for K_b therefore decreases: excited-state peaks sit at lower detected energy than the ground-state peak. That is the basis of missing-mass / reaction spectroscopy.

Bibliography

[1] R. A. Dunlap, *An Introduction to the Physics of Nuclei and Particles* (2nd ed.), IOP Publishing, Bristol (2023), DOI: 10.1088/978-0-7503-6094-4, Ch. 3 (Nuclear composition and size).

[2] W. Demtröder, *Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2023).

[3] S. Petrera, *Problems and Solutions in Nuclear and Particle Physics*, UNITEXT for Physics, Springer, Cham (2019).

[4] F. Terranova, *A Modern Primer in Particle and Nuclear Physics*, Oxford University Press (2021).

[5] S. D'Auria, *Introduction to Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2018).

[6] K. S. Krane, *Introductory Nuclear Physics*, Wiley, New York (1988).

[7] R. Hofstadter, "Electron scattering and nuclear structure," Rev. Mod. Phys. **28**, 214 (1956).

[8] H. F. Ehrenberg, R. Hofstadter, U. Meyer-Berkhout, D. G. Ravenhall, and S. E. Sobottka, "High-Energy Electron Scattering and the Charge Distribution of Carbon-12 and Oxygen-16," Phys. Rev. **113**, 666 (1959).

[9] H. Yukawa, "On the interaction of elementary particles. I," Proc. Phys.-Math. Soc. Japan **17**, 48 (1935).

[10] R. D. Woods and D. S. Saxon, "Diffuse surface optical model for nucleon-nuclei scattering," Phys. Rev. **95**, 577 (1954).

[11] H. G. J. Moseley, "The high-frequency spectra of the elements," Phil. Mag. **26**, 1024 (1913); **27**, 703 (1914).

[12] C. F. von Weizsäcker, "Zur Theorie der Kernmassen," Z. Phys. **96**, 431 (1935).

[13] P. Cappellaro, *Introduction to Applied Nuclear Physics* (MIT OpenCourseWare / LibreTexts), CC BY-NC-SA 4.0.

[14] J.-L. Basdevant, J. Rich, and M. Spiro, *Fundamentals in Nuclear Physics: From Nuclear Structure to Cosmology*, Springer, New York (2005).

[15] D. Tong, *Particle Physics*, Lecture notes (University of Cambridge / CERN school notes).

[16] M. G. Mayer, "On closed shells in nuclei. II," Phys. Rev. **75**, 1969 (1949).

[17] O. Haxel, J. H. D. Jensen, and H. E. Suess, "On the "magic numbers" in nuclear structure," Phys. Rev. **75**, 1766 (1949).

[18] T. Schmidt, "Über die magnetischen Momente der Atomkerne," Z. Phys. **106**, 358 (1937).

[19] L. W. Nordheim, "On spins, moments, and shells in nuclei," Phys. Rev. **78**, 294 (1950).

[20] Evaluated Nuclear Structure Data File (ENSDF), National Nuclear Data Center, Brookhaven National Laboratory, `https://www.nndc.bnl.gov/ensdf/`.

[21] NuDat 3, National Nuclear Data Center, Brookhaven National Laboratory, `https://www.nndc.bnl.gov/nudat3/`.

[22] N. J. Stone, "Table of nuclear magnetic dipole and electric quadrupole moments," At. Data Nucl. Data Tables **90**, 75 (2005); updated compilations at NNDC.

[23] W. N. Cottingham and D. A. Greenwood, *An Introduction to Nuclear Physics* (2nd ed.), Cambridge University Press, Cambridge (2001).

[24] H. Geiger and J. M. Nuttall, "The ranges of the α particles from the radioactive substances," Phil. Mag. **22**, 613 (1911).

[25] G. Gamow, "Zur Quantentheorie des Atomkernes," Z. Phys. **51**, 204 (1928).

[26] R. W. Gurney and E. U. Condon, "Wave mechanics and radioactive disintegration," Nature **122**, 439 (1928); Phys. Rev. **33**, 127 (1929).

[27] M. Wang, W. J. Huang, F. G. Kondev, G. Audi, and S. Naimi, "The AME 2020 atomic mass evaluation (II). Tables, graphs and references," Chin. Phys. C **45**, 030003 (2021).

[28] Laboratoire National Henri Becquerel (LNHB), Recommended decay data, `http://www.lnhb.fr/nuclear-data/nuclear-data-table/`.

[29] L. M. Langer, R. D. Moffat, and H. C. Price, Jr., "Low energy beta-ray spectra: P^{32} and Cu^{64}," Phys. Rev. **76**, 1725 (1949).

[30] L. M. Langer and H. C. Price, Jr., "Shapes of the beta-spectra of the forbidden transitions of Sb^{124}, Rb^{86}, Sr^{89}, Y^{91}, and Cs^{137}," Phys. Rev. **75**, 1769 (1949).

[31] D. C. Camp and L. M. Langer, "The shape of the ^{66}Ga positron spectrum and the contribution from $0^+ \to 0^+$ transitions," Phys. Rev. **129**, 1782 (1963).