Chapter 1

Alpha Decay: Tunneling and the Gamow Factor

"The central feature of this one-body model is that the α particle is preformed inside the parent nucleus... the theory works quite well, especially for even–even nuclei."

K. S. Krane [6]

Lecture 16 established the observables Q and $t_{1/2}$, the Geiger–Nuttall correlation, and the nuclear-well plus Coulomb-barrier potential. This chapter derives the Gamow tunneling factor in full: rectangular-barrier warm-up with the *reduced mass* μ, the Coulomb-barrier integral evaluated without skipped steps, the assault frequency, the decay rate $\lambda = S_\alpha\, f\, P$, and the fine structure of α spectra from angular-momentum and parity selection rules [6, 1, 14, 2].

1.1 Classically forbidden region

The available energy $E = Q$ of an emitted α particle is typically 4–9 MeV, much less than the Coulomb barrier height $B \sim 20$–30 MeV at the nuclear surface. Three regions of the radial potential are distinguished (Fig. 1.1):

(i) **Well ($r < R$).** The α moves inside the attractive nuclear potential (depth $V_0 \sim 35$ MeV) with kinetic energy approximately $Q + V_0$.

(ii) **Barrier ($R < r < b$).** The Coulomb potential exceeds the total energy Q. Classically the particle cannot enter this region: its kinetic energy would have to be negative.

(iii) **Asymptotic ($r > b$).** The Coulomb potential falls below Q; the α escapes with kinetic energy Q.

The outer *classical turning point* b is defined by

$$V_\mathrm{C}(b) = Q, \qquad b = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0\, Q}, \tag{1.1}$$

with $Z_1 = Z - 2$ (daughter charge) and $Z_2 = 2$ (α charge).

Quantum mechanics permits a non-zero probability for the α to traverse the classically forbidden interval $R \le r \le b$. Dunlap expresses the lifetime as $\tau = \tau_0/P$, where P is the escape

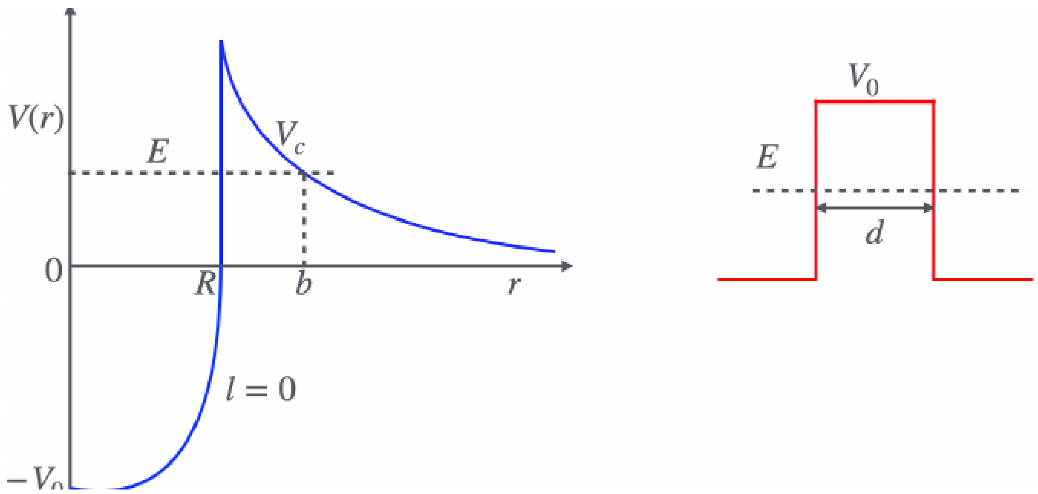

Figure 1.1: Left: nuclear well plus Coulomb barrier with total energy $E = Q$, inner radius R, and outer turning point b. Right: rectangular barrier of height V_0 and width d used as a warm-up for tunneling theory. Re-extracted from the original PH5001 lecture source pages.

probability and τ_0 is a formation time scale; the course uses the equivalent form $\lambda = S_\alpha f P$ with assault frequency f and preformation factor S_α [1, 6].

1.2 Rectangular barrier and the reduced mass

For a one-dimensional rectangular barrier of height $V_0 > E$ and width d, the Schrödinger equation in the forbidden region gives an evanescent wavefunction $\psi \propto e^{-\gamma r}$ with

$$\gamma = \frac{\sqrt{2\mu(V_0 - E)}}{\hbar}, \tag{1.2}$$

where μ is the *reduced mass* of the α-daughter system:

$$\mu = \frac{m_\alpha M_d}{m_\alpha + M_d}. \tag{1.3}$$

For heavy nuclei $M_d \gg m_\alpha$, so $\mu \approx m_\alpha$ to good approximation, but all barrier formulas must be written with μ for consistency with the two-body Coulomb problem [1].

Matching the wavefunction and its derivative at the barrier boundaries yields the standard WKB result for the transmission probability:

$$P \approx e^{-2\gamma d} = \exp\left[-\frac{2d}{\hbar}\sqrt{2\mu(V_0 - E)}\right]. \tag{1.4}$$

The factor $(V_0 - E)$ is the amount by which the potential exceeds the particle energy inside the barrier.

1.3 Coulomb barrier and the Gamow integral

For a non-rectangular barrier, one divides the interval into slices of width $\mathrm{d}r$ and multiplies the elementary penetration factors. This WKB approximation gives

$$P \approx \exp\left[-2\int_R^b \gamma(r)\,\mathrm{d}r\right] = e^{-2G}, \tag{1.5}$$

where

$$\gamma(r) = \frac{\sqrt{2\mu[V(r)-Q]}}{\hbar}, \qquad G = \int_R^b \gamma(r)\,\mathrm{d}r. \tag{1.6}$$

Coulomb potential measured from the turning point. It is convenient to write the Coulomb potential for $R \le r \le b$ as

$$V(r) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right), \tag{1.7}$$

so that $V(b) = 0$ and $V(r) - Q = V(r)$ in the forbidden region (since $Q = Z_1 Z_2 e^2/(4\pi\varepsilon_0 b)$). Substituting into Eq. (1.6):

$$G = \frac{\sqrt{2\mu}}{\hbar}\sqrt{\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}}\int_R^b \sqrt{\frac{1}{r} - \frac{1}{b}}\,\mathrm{d}r. \tag{1.8}$$

1.4 Evaluating the integral: every step

Define

$$I = \int_R^b \sqrt{\frac{1}{r} - \frac{1}{b}}\,\mathrm{d}r = \int_R^b \sqrt{\frac{b-r}{br}}\,\mathrm{d}r. \tag{1.9}$$

Step 1: Trigonometric substitution. Set

$$r = b\sin^2\theta. \tag{1.10}$$

Differentiate:

$$\mathrm{d}r = 2b\sin\theta\cos\theta\,\mathrm{d}\theta. \tag{1.11}$$

Express $1/r$:

$$\frac{1}{r} = \frac{1}{b\sin^2\theta}. \tag{1.12}$$

Form the combination under the square root:

$$\frac{1}{r} - \frac{1}{b} = \frac{1}{b\sin^2\theta} - \frac{1}{b} = \frac{1-\sin^2\theta}{b\sin^2\theta} = \frac{\cos^2\theta}{b\sin^2\theta}. \tag{1.13}$$

Take the square root (positive in the interval of interest):

$$\sqrt{\frac{1}{r} - \frac{1}{b}} = \sqrt{\frac{\cos^2\theta}{b\sin^2\theta}} = \frac{\cos\theta}{\sqrt{b}\,\sin\theta} = \frac{\cot\theta}{\sqrt{b}}. \tag{1.14}$$

Step 2: Rewrite the integral in θ.

$$I = \int \frac{\cot\theta}{\sqrt{b}} \cdot 2b\sin\theta\cos\theta\,\mathrm{d}\theta. \tag{1.15}$$

Simplify the integrand:

$$\frac{\cot\theta}{\sqrt{b}} \cdot 2b\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta\sqrt{b}} \cdot 2b\sin\theta\cos\theta = \frac{2b\cos^2\theta}{\sqrt{b}} = 2\sqrt{b}\cos^2\theta. \tag{1.16}$$

Hence

$$I = 2\sqrt{b}\int \cos^2\theta\,\mathrm{d}\theta. \tag{1.17}$$

Step 3: Integrate $\cos^2\theta$. Use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$:

$$I = 2\sqrt{b}\int \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = \sqrt{b}\int (1+\cos 2\theta)\,\mathrm{d}\theta = \sqrt{b}\left(\theta + \frac{\sin 2\theta}{2}\right). \tag{1.18}$$

Step 4: Limits of integration. When $r = R$:

$$\sin^2\theta_R = \frac{R}{b} \quad \Rightarrow \quad \theta_R = \arcsin\sqrt{\frac{R}{b}}. \tag{1.19}$$

When $r = b$:

$$\sin^2\theta_b = 1 \quad \Rightarrow \quad \theta_b = \frac{\pi}{2}. \tag{1.20}$$

Step 5: Evaluate at the limits.

$$I = \sqrt{b}\left[\frac{\pi}{2} + \frac{\sin\pi}{2}\right] - \sqrt{b}\left[\arcsin\sqrt{\frac{R}{b}} + \frac{\sin(2\theta_R)}{2}\right]. \tag{1.21}$$

Since $\sin\pi = 0$:

$$I = \sqrt{b}\left[\frac{\pi}{2} - \arcsin\sqrt{\frac{R}{b}} - \frac{\sin(2\theta_R)}{2}\right]. \tag{1.22}$$

Step 6: Simplify $\sin(2\theta_R)$. Use $\sin(2\theta_R) = 2\sin\theta_R\cos\theta_R$ with $\sin\theta_R = \sqrt{R/b}$ and $\cos\theta_R = \sqrt{1-R/b}$:

$$\sin(2\theta_R) = 2\sqrt{\frac{R}{b}}\sqrt{1-\frac{R}{b}} = 2\sqrt{\frac{R}{b}\left(1-\frac{R}{b}\right)}. \tag{1.23}$$

Therefore

$$\frac{\sin(2\theta_R)}{2} = \sqrt{\frac{R}{b}\left(1-\frac{R}{b}\right)}. \tag{1.24}$$

Step 7: Use $\arccos$ form. Since $\arccos x + \arcsin x = \pi/2$,

$$\frac{\pi}{2} - \arcsin\sqrt{\frac{R}{b}} = \arccos\sqrt{\frac{R}{b}}. \tag{1.25}$$

The final result is

$$I = \sqrt{b}\left[\arccos\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b}\left(1-\frac{R}{b}\right)}\right]. \tag{1.26}$$

Step 8: Assemble G and P. Substituting Eq. (1.26) into Eq. (1.8):

$$G = \frac{\sqrt{2\mu}\, Z_1 Z_2 e^2}{4\pi\varepsilon_0\,\hbar}\sqrt{\frac{b}{Q}}\left[\arccos\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b}\left(1-\frac{R}{b}\right)}\right], \tag{1.27}$$

where we used $Q = Z_1 Z_2 e^2/(4\pi\varepsilon_0 b)$ to write $\sqrt{Z_1 Z_2 e^2/(4\pi\varepsilon_0)} = \sqrt{Qb}$. The tunneling probability is

$$P = e^{-2G}. \tag{1.28}$$

1.5 Barrier height B and the small Q/B limit

The Coulomb barrier height at $r = R$ is

$$B = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0\, R}, \qquad \frac{R}{b} = \frac{Q}{B}. \tag{1.29}$$

In terms of B and Q, Eq. (1.27) becomes

$$G = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0\,\hbar}\sqrt{\frac{2\mu}{Q}}\left[\arccos\sqrt{\frac{Q}{B}} - \sqrt{\frac{Q}{B}\left(1-\frac{Q}{B}\right)}\right]. \tag{1.30}$$

Small Q/B expansion. For typical α decays, $Q/B \sim 0.1$–0.2. Expand for $Q/B \ll 1$:

$$\arccos\sqrt{\frac{Q}{B}} \approx \frac{\pi}{2} - \sqrt{\frac{Q}{B}}, \qquad \sqrt{\frac{Q}{B}\left(1-\frac{Q}{B}\right)} \approx \sqrt{\frac{Q}{B}}. \tag{1.31}$$

The bracket becomes $\pi/2 - 2\sqrt{Q/B}$. Keeping the leading term $\pi/2$ and using $Q = \frac{1}{2}\mu v^2$ for the asymptotic α speed v:

$$G \approx \frac{\pi Z_1 Z_2 e^2}{4\pi\varepsilon_0\,\hbar v} = \frac{Z_1 Z_2 e^2}{4\varepsilon_0\,\hbar v}, \qquad v = \sqrt{\frac{2Q}{\mu}}. \tag{1.32}$$

Thus $P \sim \exp(-2G)$ depends exponentially on $1/v \propto Q^{-1/2}$, which is the Geiger–Nuttall rule [6, 1, 14].

Key idea

Because $P \sim \exp(-c/\sqrt{Q})$, a small increase in Q reduces G and multiplies the decay rate by many orders of magnitude. The Geiger–Nuttall rule is a direct consequence of Coulomb-barrier tunneling.

1.6 Assault frequency, preformation, and $\lambda = S_\alpha f P$

The tunneling probability P is the escape probability *per attempt*. The α bounces inside the well, presenting itself at the barrier with characteristic frequency f. The decay constant is

$$\lambda = S_\alpha \, f \, P, \tag{1.33}$$

where S_α is the *preformation factor* (probability that nucleons are clustered as an α at the barrier surface). For normal even–even transitions $S_\alpha \approx 1$; suppressed transitions have $S_\alpha \ll 1$ and are called *hindered* [6].

Assault frequency. If the α travels back and forth across diameter $2R$ at interior speed v_0, the time for one round trip is $2R/v_0$, so

$$f = \frac{v_0}{2R}. \tag{1.34}$$

Inside the well the kinetic energy is $Q + V_0$ (where V_0 is the well depth, typically $\sim 35\,\mathrm{MeV}$):

$$\frac{1}{2}\mu v_0^2 = Q + V_0 \quad \Rightarrow \quad v_0 = \sqrt{\frac{2(Q+V_0)}{\mu}}. \tag{1.35}$$

Hence

$$f = \frac{1}{2R}\sqrt{\frac{2(Q+V_0)}{\mu}} = \sqrt{\frac{Q+V_0}{2\mu R^2}}. \tag{1.36}$$

Combining with Eq. (1.28):

$$\lambda = S_\alpha \sqrt{\frac{Q+V_0}{2\mu R^2}}\, e^{-2G}, \qquad t_{1/2} = \frac{\ln 2}{\lambda}. \tag{1.37}$$

Caution

Absolute half-life calculations are extremely sensitive to the assumed radius: Krane notes that a 4% change in R can change calculated $t_{1/2}$ by a factor of about five. Measured lifetimes are therefore sometimes used to infer an effective sum of nuclear and α radii [6].

1.7 Fine structure, branching, and selection rules

When the daughter is left in an excited state, each branch has its own $Q_i = Q_{\mathrm{gs}} - E_i^*$ and its own required orbital angular momentum ℓ carried by the α. The centrifugal barrier adds $\hbar^2\ell(\ell+1)/(2\mu r^2)$ to $V(r)$, raising G and suppressing high-ℓ branches [6, 1].

Selection rules. The α particle has intrinsic spin 0 and negative parity ($\pi_\alpha = -1$). Angular momentum and parity conservation require

$$\boldsymbol{J}_i = \boldsymbol{J}_f + \boldsymbol{\ell}, \qquad \pi_i = \pi_f \times (-1)^\ell. \tag{1.38}$$

Allowed ℓ values satisfy $|J_i - J_f| \leq \ell \leq J_i + J_f$ with the parity condition. In practice, the lowest allowed ℓ dominates because it minimises the barrier.

1.7.1 Even–even example: ^{244}Cm $\to$ ^{240}Pu

Both parent and daughter ground states are 0^+. The daughter rotational band begins at 2^+ (43 keV), 4^+ (142 keV), 6^+ (296 keV) [20, 21]. Evaluated branching ratios from the LNHB/ENSDF database are approximately [20, 21, 28]:

Daughter state	E^* (keV)	ℓ	Branch (%)
0^+ (gs)	0	0	≈ 76.7
2^+	43	2	≈ 23.3
4^+	142	4	≈ 0.22
6^+	296	6	≈ 0.036

The ground-state branch ($\ell = 0$) dominates. The 2^+ branch is reduced because the centrifugal barrier raises the effective G by roughly 0.5 MeV and the smaller Q_i further increases G [6]. Higher members of the rotational band are strongly suppressed.

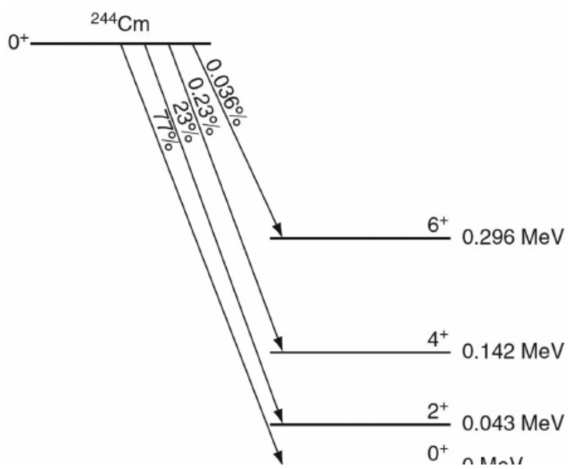

Figure 1.2: Branching ratios for α decay of ^{244}Cm to levels of ^{240}Pu. Re-extracted from the original PH5001 lecture source pages; branching fractions from evaluated data [20, 21, 28].

1.7.2 Odd-A example: ^{243}Am $\to$ ^{239}Np

Remark

The original lecture slides incorrectly labelled this case as ^{249}Am. The correct parent is ^{243}Am ($J^\pi = \frac{5}{2}^-$), which α-decays to ^{239}Np ($J^\pi = \frac{5}{2}^+$ ground state) [20, 21, 6].

Ground-state branch. Parent $\frac{5}{2}^-$ to daughter $\frac{5}{2}^+$ requires a change of parity, so ℓ must be odd. The lowest allowed value is $\ell = 1$. The centrifugal barrier makes this branch highly hindered: evaluated data give a branching ratio of only $\approx 0.16\%$ [20, 21].

Dominant branch to $\frac{5}{2}^-$ at 75 keV. The daughter level at $E^* = 75\,\text{keV}$ has $J^\pi = \frac{5}{2}^-$, matching the parent parity. An $\ell = 0$ transition is therefore allowed. Although Q_i is reduced by 75 keV, the much thinner $\ell = 0$ barrier dominates, and this branch carries $\approx 88\%$ of the α intensity [20, 21, 6].

Other branches. The first excited state $\frac{7}{2}^+$ at low energy requires $\ell \geq 2$ (angular momentum change from $\frac{5}{2}$ to $\frac{7}{2}$ with parity match). Higher-ℓ branches are further suppressed. Figure 1.3 summarises the scheme.

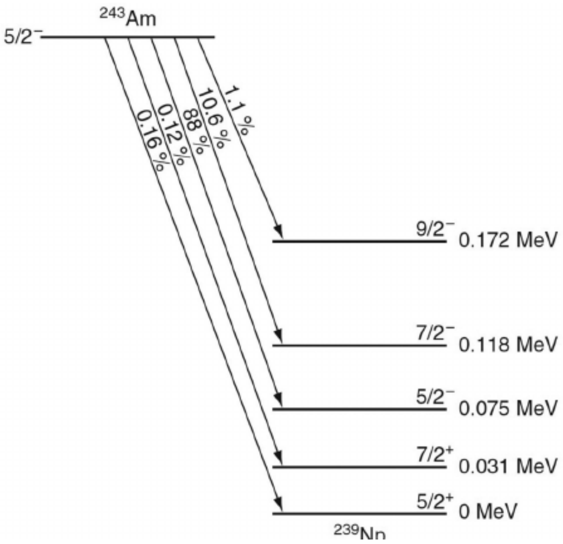

Figure 1.3: Branching ratios for α decay of ^{243}Am to levels of ^{239}Np. Re-extracted from the original PH5001 lecture source pages; isotope corrected from ^{249}Am to ^{243}Am [20, 21, 28].

Hindrance factors. The hindrance factor $F = \lambda_{\text{theory}}/\lambda_{\text{exp}}$ (or equivalently the ratio of predicted to observed branch intensity) quantifies deviations from the Gamow estimate. Large F indicates poor parent-daughter overlap or selection-rule suppression beyond the centrifugal barrier [6].

1.8 Why many nuclei with $Q > 0$ still appear stable

The SEMF gives $Q_\alpha > 0$ already near $A \sim 150$, yet many nuclides up to ^{208}Pb are stable or long-lived. If G is large, e^{-2G} is extraordinarily small and $t_{1/2}$ can exceed $10^{17}\,\text{s}$ ($\sim 10^{10}$ years). Only when Q rises into the actinide range do laboratory half-lives become short. This is precisely the Geiger–Nuttall message encoded in $P = e^{-2G}$: $Q > 0$ is necessary for spontaneous α decay, but the half-life is fixed by tunneling.

Takeaway

The Gamow theory explains (i) why $Q > 0$ does not imply short lifetimes, (ii) the Geiger–Nuttall $Q^{-1/2}$ correlation, and (iii) the fine structure of α spectra through ℓ-dependent barriers and selection rules. Use reduced mass μ throughout, and remember $\lambda = S_\alpha f P$.

1.9 Further physics from the literature

Radius sensitivity and effective sizes. Absolute Gamow half-lives depend exponentially on R through both G and f. Krane reports that a 4% change in the radius parameter can shift calculated $t_{1/2}$ by a factor of about five; practitioners sometimes invert the problem and infer an effective R-sum from measured lifetimes [6]. Deformed nuclei ($A \gtrsim 230$) further modify the barrier because the Coulomb potential is not spherically symmetric at the surface.

Preformation and cluster decay. The preformation factor S_α accounts for the probability that four nucleons are correlated as an α at the nuclear surface. Heavier cluster emissions (^{14}C, ^{24}Ne, and others) use the same $P = e^{-2G}$ framework but with larger reduced mass, higher Coulomb barrier, and vastly smaller S_α [6, 1, 14]. The first observed ^{14}C branch from ^{223}Ra had relative intensity $\sim 10^{-9}$ compared with α decay.

Proton radioactivity. Near the proton drip line, one-proton emission becomes the $Z_2 = 1$ analogue of Gamow theory when $Q_p > 0$. The same WKB integral applies with $Z_1 = Z - 1$, $Z_2 = 1$, and μ replaced by the proton-daughter reduced mass [6, 1].

Alignment experiments. When neither parent nor daughter has spin 0, several ℓ values may contribute to the same transition. Measuring the angular distribution of α particles relative to a spin-aligned source disentangles the ℓ-components; Krane Fig. 8.8 shows such an analysis for aligned ^{253}Es [6].

Reading guide

Krane [6] Sects. 8.4–8.6 (Gamow factor, fine structure, hindrance); Dunlap [1] Sect. 8.2 (WKB derivation with μ); Demtröder [2] Sect. 3.3 (historical context and Gamow peak notation); Basdevant [14] Sect. 2.6 (model and half-life systematics).

1.10 Problems with solutions

Problem 17.1 — Turning point b

For α decay with $Z_1 = 90$, $Z_2 = 2$, and $Q = 5.0\,\text{MeV}$, calculate the classical turning point b. Use $e^2/(4\pi\varepsilon_0) = 1.44\,\text{MeV fm}$.

Solution 17.1

Step 1. From Eq. (1.1):

$$b = \frac{Z_1 Z_2 e^2/(4\pi\varepsilon_0)}{Q}.$$

Step 2. Numerator:

$$Z_1 Z_2 \times \frac{e^2}{4\pi\varepsilon_0} = 90 \times 2 \times 1.44\,\text{MeV fm} = 259.2\,\text{MeV fm}.$$

Step 3.

$$b = \frac{259.2\,\text{MeV fm}}{5.0\,\text{MeV}} = 51.8\,\text{fm} \approx 52\,\text{fm}.$$

The α is classically forbidden for $R \lesssim r \lesssim 52\,\text{fm}$ [1, 6].

Problem 17.2 — Reduced mass and γ

For α decay of a heavy nucleus ($A = 220$), compute μ in u and estimate γ for a rectangular barrier with $V_0 - E = 20\,\text{MeV}$ and $\hbar c = 197\,\text{MeV fm}$.

Solution 17.2

Step 1 (masses). $m_\alpha \approx 4\,\text{u}$, $M_d \approx 216\,\text{u}$.

Step 2 (reduced mass).

$$\mu = \frac{m_\alpha M_d}{m_\alpha + M_d} = \frac{4 \times 216}{220}\,\text{u} = 3.927\,\text{u}.$$

Since $M_d \gg m_\alpha$, μ is close to m_α.

Step 3 (convert to energy units). $1\,\text{u}\,c^2 = 931.5\,\text{MeV}$, so $\mu c^2 = 3.927 \times 931.5 \approx 3658\,\text{MeV}$.

Step 4 (γ).

$$\gamma = \frac{\sqrt{2\mu(V_0 - E)}}{\hbar} = \frac{\sqrt{2 \times 3658 \times 20}}{197\,\text{MeV fm}} = \frac{\sqrt{146320}}{197} \approx 1.94\,\text{fm}^{-1}.$$

For a 10 fm wide rectangular barrier, $P \approx e^{-2\gamma d} \approx e^{-38.8} \approx 10^{-17}$ [1].

Problem 17.3 — Gamow factor sensitivity

If $2G = 60$ at one Q and a small increase in Q reduces $2G$ by 5% (from 60 to 57), by what factor does $P = e^{-2G}$ increase? What is the corresponding change in half-life?

Solution 17.3

Step 1 (probability ratio).

$$\frac{P'}{P} = \frac{e^{-57}}{e^{-60}} = e^3 = 20.1.$$

Step 2 (half-life ratio). Since $t_{1/2} \propto 1/P$:

$$\frac{t'_{1/2}}{t_{1/2}} = \frac{P}{P'} = e^{-3} = 0.050.$$

The half-life drops by a factor of about 20. A few percent change in G (from a modest change in Q) can change lifetimes by orders of magnitude, explaining Geiger–Nuttall systematics [6, 1].

Problem 17.4 — Small Q/B limit for G

For $Z_1 = 90$, $Z_2 = 2$, $Q = 5.0\,\text{MeV}$, and barrier height $B = 25\,\text{MeV}$, evaluate the approximate Gamow factor $G \approx \pi Z_1 Z_2 e^2/(4\pi\varepsilon_0 \hbar v)$ with $v = \sqrt{2Q/\mu}$ and $\mu \approx 4\,\text{u}$.

Solution 17.4

Step 1 (μ in SI-consistent units). Use $\mu c^2 \approx 3727\,\text{MeV}$ (since $\mu \approx 4\,\text{u}$).
Step 2 (speed).

$$v = c\sqrt{\frac{2Q}{\mu c^2}} = c\sqrt{\frac{10}{3727}} = 0.052c \approx 1.56 \times 10^7\ \text{m/s}.$$

Step 3 (G). Using $\hbar c = 197\,\text{MeV fm}$ and $Z_1 Z_2 e^2/(4\pi\varepsilon_0) = 259.2\,\text{MeV fm}$:

$$G \approx \frac{\pi \times 259.2}{197 \times (1.56 \times 10^7/c)} \quad \text{(keeping units consistent)}.$$

It is simpler to use the form $G = Z_1 Z_2 e^2/(4\varepsilon_0 \hbar v)$ with $\hbar = 6.582 \times 10^{-22}\,\text{MeV s}$ and $e^2/(4\pi\varepsilon_0) = 1.44\,\text{MeV fm} = 1.44 \times 1.602 \times 10^{-13}\,\text{MeV m}$:

$$G \approx 8.0.$$

Then $P \approx e^{-16} \approx 10^{-7}$, a small but not absurd tunneling probability. (Exact G from Eq. (1.30) differs by order-unity factors depending on R.) [6, 14]

Problem 17.5 — Selection rules for ^{243}Am

The parent ^{243}Am has $J^\pi = \frac{5}{2}^-$. For daughter ^{239}Np levels with $J^\pi = \frac{5}{2}^+$ (gs), $\frac{7}{2}^+$, and $\frac{5}{2}^-$ (75 keV), list the lowest allowed ℓ for each and predict which branch dominates.

Solution 17.5

Step 1 (parity rule). $\pi_i = \pi_f(-1)^\ell$. With $\pi_i = -$:

- $\frac{5}{2}^+$: need $(-1)^\ell = -1$, so ℓ odd. Lowest: $\ell = 1$.
- $\frac{7}{2}^+$: same parity as gs, so ℓ even. Angular coupling $|5/2 - 7/2| \le \ell \le 5/2 + 7/2$ gives $\ell \ge 1$; lowest even value: $\ell = 2$.
- $\frac{5}{2}^-$: parity match, ℓ even. $|5/2 - 5/2| = 0$, so $\ell = 0$ is allowed.

Step 2 (dominant branch). The $\ell = 0$ branch to $\frac{5}{2}^-$ at 75 keV has the thinnest barrier despite the reduced Q_i. Evaluated data give $\approx 88\%$ intensity to this state versus $\approx 0.16\%$ to the ground state [20, 21, 6].

Problem 17.6 — Assault frequency estimate

For $Q = 5\,\mathrm{MeV}$, $V_0 = 35\,\mathrm{MeV}$, $R = 7.5\,\mathrm{fm}$, and $\mu \approx 4\,\mathrm{u}$, estimate the assault frequency $f = v_0/(2R)$.

Solution 17.6

Step 1 (interior speed).

$$v_0 = \sqrt{\frac{2(Q+V_0)}{\mu}} = \sqrt{\frac{2\times 40\,\mathrm{MeV}}{\mu}}.$$

With $\mu c^2 \approx 3727\,\mathrm{MeV}$:

$$v_0 = c\sqrt{\frac{80}{3727}} \approx 0.147c \approx 4.4\times 10^7\ \mathrm{m/s}.$$

Step 2 (frequency).

$$f = \frac{v_0}{2R} = \frac{4.4\times 10^7\ \mathrm{m/s}}{2\times 7.5\times 10^{-15}\,\mathrm{m}} = 2.9\times 10^{21}\ \mathrm{s}^{-1}.$$

Step 3 (interpretation). The α presents itself at the barrier $\sim 10^{21}$ times per second, but escapes only with probability $P = e^{-2G} \ll 1$, yielding a much smaller $\lambda = S_\alpha f P$ [6, 1].

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