Chapter 1

Shell-Model Excited States and Collective Motion

"Experimentally, it is observed that nuclei can have several excited states… The nuclear shell model is instrumental in explaining these excited states."

lecture notes, PH5001

Ground-state I^π and Schmidt moments test the filling of the mean field. Excited states test whether the same orbitals, with particle promotions and pair breaking, organise the low-lying spectrum. Near closed shells the extreme shell model works well (classic example: ^{17}O). In mid-shell even–even nuclei the first 2^+ state is collective, and vibrational or rotational models are required [6, 1, 14].

1.1 Ground state of ^{17}O

Oxygen-17 has $Z = 8$, $N = 9$. Protons fill the closed shell through $1p_{1/2}$ (magic $Z = 8$) and contribute $I = 0$. Neutrons fill

$$(1s_{1/2})^2\,(1p_{3/2})^4\,(1p_{1/2})^2\,(1d_{5/2})^1. \tag{1.1}$$

The unpaired neutron is in $1d_{5/2}$ ($\ell = 2$, $j = \frac{5}{2}$), so

$$I^\pi = \tfrac{5}{2}^+, \tag{1.2}$$

matching experiment. The closed proton shell and nearly closed neutron p shell make ^{17}O a textbook single-particle nucleus: the core is ^{16}O and the valence neutron is weakly coupled to it.

1.2 Single-particle excitations of ^{17}O

Promoting the valence neutron to a higher empty orbit changes I^π without breaking the ^{16}O core:

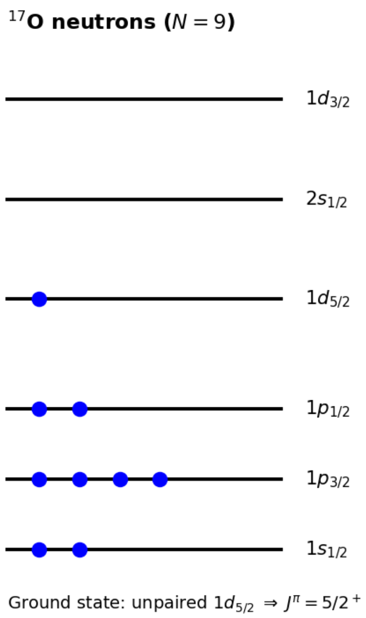

Figure 1.1: Neutron filling for ^{17}O. The ground state is fixed by the unpaired $1d_{5/2}$ neutron (adapted from the original PH5001 lecture figure).

Configuration (valence n)	I^π	Character
$1d_{5/2}$	$\frac{5}{2}^+$	ground state
$2s_{1/2}$	$\frac{1}{2}^+$	first s-wave excitation
$1d_{3/2}$	$\frac{3}{2}^+$	spin–orbit partner of $d_{5/2}$

These positive-parity states appear in the low-lying spectrum and carry large single-particle strength in one-neutron transfer. Evaluated levels place the $1/2^+$ state at 871 keV and show additional positive- and negative-parity excitations consistent with *sd* and *p*-shell rearrangements [20, 21, 6].

1.3 Particle–hole excitations in ^{17}O

Higher states involve exciting a nucleon from a filled orbit, leaving a hole.

Hole in $1p_{3/2}$. Promote one neutron from $1p_{3/2}$ into $1d_{5/2}$. After the transition one has three neutrons in $1p_{3/2}$ (odd occupancy) and two in $1d_{5/2}$. The unpaired hole in $1p_{3/2}$ yields

$$I^\pi = \frac{3}{2}^- \tag{1.3}$$

($\ell = 1 \Rightarrow \pi = -$).

Hole in $1p_{1/2}$. Promote from $1p_{1/2}$ into $1d_{5/2}$: one unpaired neutron remains in $1p_{1/2}$, giving

$$I^\pi = \frac{1}{2}^-. \tag{1.4}$$

Multi-particle couplings. When three unpaired neutrons sit in different orbits (e.g. $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$), their angular momenta couple to several total I. Parity of a multi-particle configuration is the product of single-particle parities,

$$\pi = \prod_i (-1)^{\ell_i}. \tag{1.5}$$

A state such as $\frac{5}{2}^-$ can arise from coupling three unequal j values with overall negative parity. The extreme model still provides the language of orbits and parities even when residual interactions fix the detailed multiplet ordering.

1.4 Even–even example: ^{38}Ar

Argon-38 has $Z = 18$, $N = 20$. Neutrons fill the magic $N = 20$ shell; protons sit two holes below $Z = 20$. The ground state is even–even, so

$$I^\pi = 0^+. \tag{1.6}$$

Because $N = 20$ is closed, low-lying excitations often rearrange protons.

Proton $1d_{3/2} \to 1f_{7/2}$. Leaving one proton in $1d_{3/2}$ and one in $1f_{7/2}$ produces two unequal j values. Allowed total angular momenta run from $|7/2 - 3/2| = 2$ to $7/2 + 3/2 = 5$:

$$I = 2, 3, 4, 5, \qquad \pi = (-1)^2(-1)^3 = -. \tag{1.7}$$

States $2^-, 3^-, 4^-, 5^-$ are therefore expected; the observed 3^- is naturally associated with this particle–hole (or two-proton) excitation across the *sd*–*pf* interface [6].

Proton $2s_{1/2} \to 1d_{3/2}$. Breaking a $2s_{1/2}$ pair and promoting one proton toward $1d_{3/2}$ yields $I = 1, 2$ with positive parity ($\pi = (-1)^{0+2} = +$). Forming such a state costs both pair-breaking energy ($\sim 2\,\mathrm{MeV}$) and single-particle promotion energy, so the associated 2^+ tends to lie near $\sim 4\,\mathrm{MeV}$, higher than the first collective 2^+.

1.5 Systematics of the lowest 2^+ in even–even nuclei

In the great majority of even–even nuclei the lowest excited state is 2^+. Important exceptions occur near doubly closed shells, where a 0^+ or 3^- state may intervene; globally, however, a low-lying 2^+ cannot be explained by single-particle counting alone: the same pattern appears for hundreds of nuclei over a wide range of A (Fig. 1.2).

Empirical systematics of the lowest $E(2^+)$ show three regimes [6, 1, 14]:

1. **Magic and near-magic nuclei:** $E(2^+)$ is large (several MeV), reflecting rigid spherical cores.

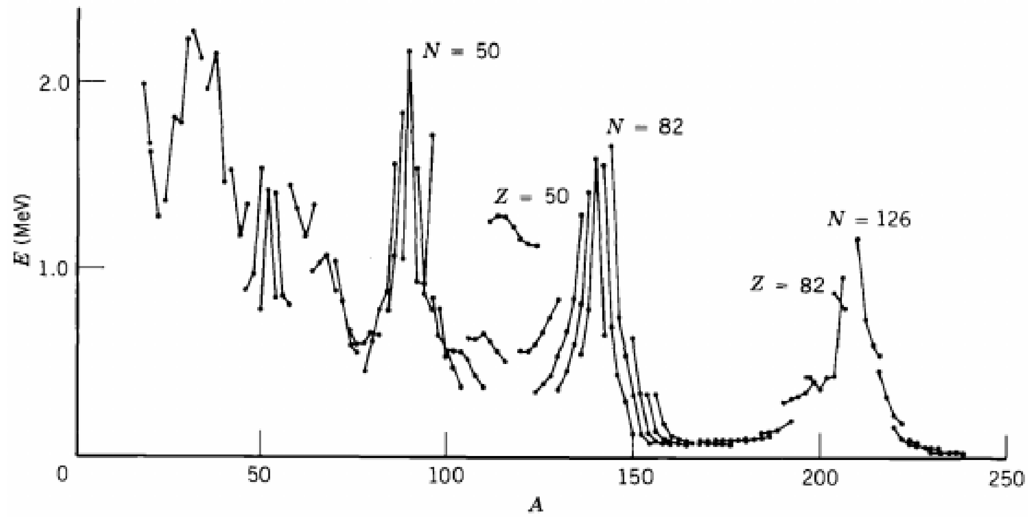

Figure 1.2: Energies of the lowest 2^+ states of even-Z, even-N nuclei versus A. Peaks occur at shell closures; mid-shell regions show much lower $E(2^+)$ (re-extracted with complete axes from the full original PH5001 source page; cf. Krane [6]).

2. **Spherical and weakly collective regions:** typical $E(2^+)$ values are of order 1 MeV. An approximately harmonic quadrupole vibrator has $E(4^+)/E(2^+) \approx 2$, while transitional nuclei need not obey this ideal ratio.

3. **Deformed regions $A \sim 150$–190 and $A \gtrsim 230$:** $E(2^+)$ drops to tens of keV, with $E(4^+)/E(2^+) \approx 10/3 \approx 3.33$, the rigid-rotor ratio.

Peaks of $E(2^+)$ at magic N or Z reconnect these collective trends to shell structure: closed shells stiffen the nucleus against deformation.

Key idea

Odd-A neighbours of closed shells: single-particle and particle–hole excitations. Even–even mid-shell nuclei: collective 2^+ (vibration or rotation). The shell model alone is incomplete for the latter.

1.6 Vibrational model

For spherical even–even nuclei ($A \lesssim 150$) the equilibrium shape is spherical. Small excess energy can excite volume-conserving surface oscillations. The nuclear radius is expanded in spherical harmonics,

$$R(\theta, \phi, t) = \bar{R}\left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_{\lambda\mu}(t)\, Y_{\lambda\mu}(\theta, \phi)\right], \tag{1.8}$$

with multipolarity λ and deformation amplitudes $\alpha_{\lambda\mu}(t)$. To derive the volume constraint, write the volume enclosed by the surface as

$$\mathcal{V} = \frac{1}{3} \int R^3(\theta, \phi, t)\, \mathrm{d}\Omega. \tag{1.9}$$

For small amplitudes, $(1+\epsilon)^3 = 1 + 3\epsilon + \mathcal{O}(\epsilon^2)$, so Eq. (1.8) gives

$$\begin{aligned}\mathcal{V} &= \frac{\bar{R}^3}{3} \int \left[1 + 3 \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}\right] \mathrm{d}\Omega + \mathcal{O}(\alpha^2) \\ &= \frac{4\pi \bar{R}^3}{3} \left(1 + \frac{3\alpha_{00}}{\sqrt{4\pi}}\right) + \mathcal{O}(\alpha^2),\end{aligned} \tag{1.10}$$

where $\int Y_{\lambda\mu} \mathrm{d}\Omega = \sqrt{4\pi}\, \delta_{\lambda 0} \delta_{\mu 0}$. Keeping $\mathcal{V} = 4\pi \bar{R}^3/3$ therefore requires

$$\alpha_{00} = 0 \qquad \text{to first order.} \tag{1.11}$$

At second order a small monopole correction compensates the volume change from $|\alpha_{\lambda\mu}|^2$. The intrinsic parity of a one-phonon excitation of multipolarity λ is $(-1)^\lambda$, because $Y_{\lambda\mu}(-\hat{r}) = (-1)^\lambda Y_{\lambda\mu}(\hat{r})$.

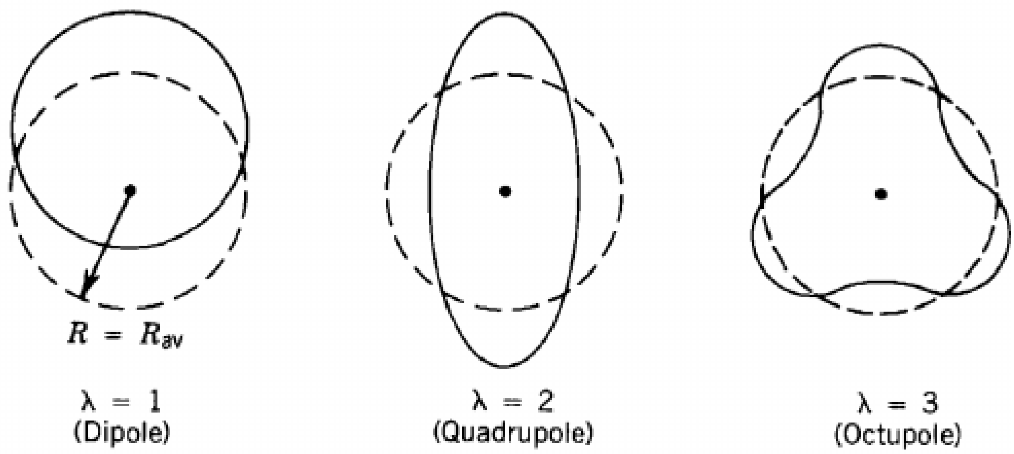

Figure 1.3: Lowest vibrational modes of a nucleus: dipole ($\lambda = 1$), quadrupole ($\lambda = 2$), octupole ($\lambda = 3$) (re-extracted from the full original PH5001 source page; cf. Krane [6]).

Breathing mode ($\lambda = 0$). With $Y_{00} = 1/\sqrt{4\pi}$,

$$R(\theta, \phi, t) = \bar{R}\big(1 + \alpha_{00}(t)\big). \tag{1.12}$$

The nucleus expands and contracts uniformly without changing shape. This isoscalar monopole mode lies at high excitation energy and does not produce the low-lying quantized shape spectrum of interest here.

Dipole mode ($\lambda = 1$). A pure $\lambda = 1$ surface deformation is equivalent to a displacement of the nuclear centre of mass. To see this explicitly, translate a sphere by a small vector $\boldsymbol{d}$. Along direction $\hat{r}$, its new surface is

$$R'(\hat{r}) = \bar{R} + \boldsymbol{d} \cdot \hat{r} + \mathcal{O}(d^2/\bar{R}). \tag{1.13}$$

The angular function $\boldsymbol{d} \cdot \hat{r}$ is a linear combination of the three $Y_{1\mu}$. Thus a $\lambda = 1$ deformation merely moves the whole sphere and can be removed by choosing the centre-of-mass origin. It is not an intrinsic excitation. Low-lying shape phonons begin at quadrupole order. (Giant dipole resonances are a different, high-energy isovector phenomenon in which protons and neutrons oscillate against each other.)

Quadrupole mode ($\lambda = 2$). The nucleus oscillates between prolate and oblate ellipsoidal shapes. This mode produces the dominant low-lying vibrational spectrum of spherical even–even nuclei.

Octupole mode ($\lambda = 3$). More complex pear-shaped surface oscillations; the one-phonon state is 3^-.

1.6.1 Phonons and the vibrational spectrum

Each multipole amplitude $\alpha_{\lambda\mu}$ behaves as a harmonic oscillator coordinate. For one normal coordinate the Hamiltonian has the form

$$H_{\lambda\mu} = \frac{\pi_{\lambda\mu}^2}{2B_\lambda} + \frac{1}{2}C_\lambda \alpha_{\lambda\mu}^2, \qquad \omega_\lambda = \sqrt{\frac{C_\lambda}{B_\lambda}}, \tag{1.14}$$

where B_λ is the collective inertia and C_λ the restoring-force constant. Canonical quantisation gives

$$E_{n_\lambda} = \left(n_\lambda + \tfrac{1}{2}\right)\hbar\omega_\lambda. \tag{1.15}$$

Only excitation energies are observed, so the common zero-point term is subtracted and n_λ quanta cost $n_\lambda \hbar\omega_\lambda$. These quanta are called **phonons**. The lowest vibrational excitation is one quadrupole phonon:

$$E = \hbar\omega_2, \qquad I^\pi = 2^+, \tag{1.16}$$

because the phonon carries angular momentum $\lambda = 2$ and parity $(-1)^\lambda = +$. For even–even nuclei the ground state is 0^+ and this 2^+ is the first excited state of the vibrational model.

Two quadrupole phonons have energy $2\hbar\omega_2$. Coupling two identical $I = 2$ bosons allows only the totally symmetric states $I = 0, 2, 4$ (the odd values $1, 3$ are forbidden for identical phonons), all with positive parity: the two-phonon triplet $0^+, 2^+, 4^+$.

Three quadrupole phonons give energy $3\hbar\omega_2$ and allowed $I^\pi = 0^+, 2^+, 3^+, 4^+, 6^+$. One octupole phonon gives

$$E = \hbar\omega_3, \qquad I^\pi = 3^-. \tag{1.17}$$

In the harmonic limit the two-phonon 4^+ and the one-phonon 2^+ satisfy

$$\frac{E(4^+)}{E(2^+)} = \frac{2\hbar\omega_2}{\hbar\omega_2} = 2, \tag{1.18}$$

matching the empirical ratio for many nuclei with $A \sim 30$–150. The equality is an ideal harmonic limit, not an exact universal rule: anharmonicities split the two-phonon multiplet and shift the ratio away from 2. For an incompressible liquid drop with surface tension σ and density ρ, the

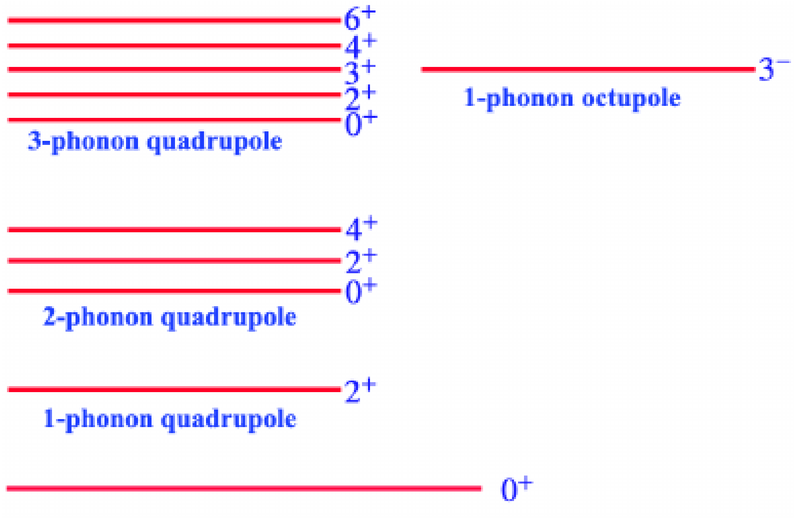

Figure 1.4: Schematic vibrational spectrum of an even–even nucleus: one-, two- and three-phonon quadrupole multiplets and the one-phonon octupole 3^- (re-extracted from the full original PH5001 source page).

classical surface-mode frequencies scale as

$$\omega_\lambda = \sqrt{\frac{\sigma(\lambda-1)(\lambda+2)\lambda}{\rho\,\bar{R}^3}}, \tag{1.19}$$

so higher multipoles are stiffer and lie higher in energy [6, 1, 14].

1.7 Rotational model

Nuclei with large static quadrupole moments (notably $A \sim 150$–190 and $A \gtrsim 230$) are permanently deformed (Fig. 1.5). For an axially symmetric rotor the intrinsic deformation may be parametrised by a quadrupole parameter β_2 through

$$R(\theta) = \bar{R}(1 + \beta_2 Y_{20}(\theta) + \cdots). \tag{1.20}$$

Rotation about an axis perpendicular to the symmetry axis produces a quantized rotational band.

Classically $E = \frac{1}{2}\mathcal{I}\omega^2 = L^2/(2\mathcal{I})$. In quantum mechanics the squared angular momentum of the rotor is replaced by $I(I+1)\hbar^2$, so

$$E_I = \frac{\hbar^2}{2\mathcal{I}}\,I(I+1). \tag{1.21}$$

For the ground band of an even–even axial nucleus the projection of angular momentum on the symmetry axis vanishes ($K = 0$). Symmetry under a rotation by π about an axis perpendicular to the symmetry axis then admits only even I with positive parity:

$$I^\pi = 0^+, 2^+, 4^+, 6^+, \ldots \tag{1.22}$$

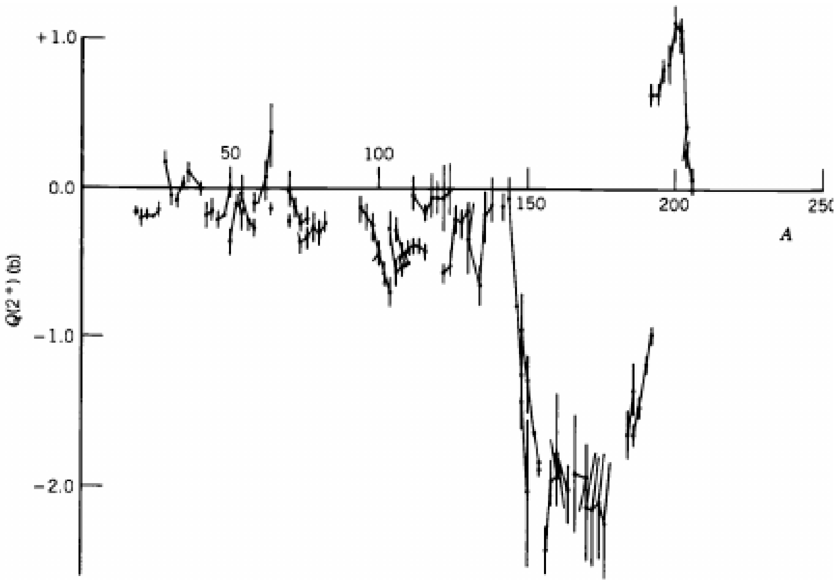

Figure 1.5: Electric quadrupole moments of the lowest 2^+ states of even–even nuclei. Large $|Q|$ in deformed regions signals permanent deformation (re-extracted with complete axes from the full original PH5001 source page; cf. Krane [6]).

(odd-I members are excluded). Using $E(2^+) = 6\hbar^2/(2\mathcal{I})$ and $E(4^+) = 20\hbar^2/(2\mathcal{I})$ one finds

$$\frac{E(4^+)}{E(2^+)} = \frac{20}{6} = \frac{10}{3} \approx 3.33, \tag{1.23}$$

the classic rotor ratio. Rotational 2^+ energies are typically of order 10–100 keV, much lower than vibrational $\sim 1\,\mathrm{MeV}$ levels [6, 14, 1].

The same deformation that produces the rotational band also generates a large intrinsic quadrupole moment Q_0. For a uniformly charged, slightly deformed nucleus,

$$Q_0 \simeq \frac{3}{\sqrt{5\pi}}\, Ze\bar{R}^2\beta_2 \qquad \text{(to first order in } \beta_2\text{)}. \tag{1.24}$$

The laboratory spectroscopic quadrupole moment of a band member is

$$Q_s(I,K) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)}\, Q_0. \tag{1.25}$$

For the $K = 0$ ground-state band this is negative when $Q_0 > 0$: a prolate intrinsic shape does not imply a positive laboratory Q_s. The reduced transition probability is

$$B(E2; IK \to I-2, K) = \frac{5}{16\pi} Q_0^2 \left|\langle IK\, 2\, 0 | I-2, K\rangle\right|^2. \tag{1.26}$$

Thus Q_s is linear in Q_0, whereas $B(E2)$ is quadratic in Q_0. Enhanced $B(E2)$ values (many Weisskopf units) are a hallmark of collective rotation, in contrast to single-particle estimates near closed shells [6, 1].

1.8 Limitations of the extreme shell model

The extreme single-particle picture fails when:

- several valence nucleons occupy an open shell (need configuration mixing and residual two-body matrix elements);
- the nucleus is soft or deformed far from magic numbers (collective rotational/vibrational degrees of freedom);
- continuum and cluster structures appear near drip lines.

Modern shell-model calculations keep the mean-field orbitals but diagonalise a residual interaction in a truncated valence space. Collective models build on the same single-particle structure: magic numbers still mark where sphericity is preferred.

Key idea

Near closed shells, excited states are particle promotions and particle–hole excitations on top of a magic core. ^{17}O is the cleanest example. Mid-shell even–even spectra demand collective vibration ($E(4^+)/E(2^+) \approx 2$) or rotation ($E(4^+)/E(2^+) \approx 3.33$).

1.9 Further physics: spectroscopic factors

One-nucleon transfer reactions such as (d, p) or (p, d) measure how much of a pure single-particle orbital is present in a nuclear state. The spectroscopic factor S quantifies that overlap: $S = 1$ for an ideal single-particle state on an inert closed core, while $S < 1$ signals fragmentation of the single-particle strength among several eigenstates [6, 1].

Near doubly magic cores, low-lying odd-A states often carry large spectroscopic factors, consistent with the extreme shell model. As one moves away from closed shells, residual interactions spread the strength and S decreases. Sum rules constrain the total strength available in a given orbit: the summed spectroscopic factors for adding or removing a nucleon from a closed shell cannot exceed the orbit's degeneracy. Fragmentation of spectroscopic strength is therefore direct experimental evidence for the limitations of the extreme independent-particle picture and for the need for configuration mixing or collective degrees of freedom.

1.10 Problems with solutions

Problem 15.1 — First excited state of ^{17}O

In the extreme single-particle model, what is I^π if the valence neutron of ^{17}O is promoted from $1d_{5/2}$ to $2s_{1/2}$? Compare with the evaluated excitation energy.

Solution 15.1

$2s_{1/2}$ has $\ell = 0$, $j = \frac{1}{2}$, so $I^\pi = \frac{1}{2}^+$. ENSDF places this state at 871 keV, matching the observed first excited state [20, 21, 6].

Problem 15.2 — ^{38}Ar multiplet

A proton is excited from $1d_{3/2}$ to $1f_{7/2}$ in ^{38}Ar. List the allowed I^{π} values for the two unpaired protons.

Solution 15.2

$j_1 = \frac{3}{2}$, $j_2 = \frac{7}{2}$ give

$$|j_1 - j_2| \leq I \leq j_1 + j_2 \quad \Rightarrow \quad I = 2, 3, 4, 5.$$

Parity $(-1)^{2+3} = -$, so $2^-, 3^-, 4^-, 5^-$ [6].

Problem 15.3 — Rotor vs vibrator ratio

Compute $E(4^+)/E(2^+)$ for a pure axial rotor with $E(I) \propto I(I+1)$. Compare with the ideal harmonic vibrator value.

Solution 15.3

For the rotor,

$$E(2) \propto 2 \cdot 3 = 6, \qquad E(4) \propto 4 \cdot 5 = 20,$$

so the ratio is $20/6 = 10/3 \approx 3.33$. For a harmonic quadrupole vibrator, $E(4^+) = 2\hbar\omega_2$ while $E(2^+) = \hbar\omega_2$, so the ratio is 2 [6, 14].

Problem 15.4 — Why $\lambda = 1$ is not a shape phonon

Explain why a pure $\lambda = 1$ term in the surface expansion does not generate an intrinsic low-lying vibrational excitation.

Solution 15.4

A $\lambda = 1$ deformation displaces the nuclear surface in a way that is equivalent to a shift of the centre of mass. Transforming to the CM frame removes the amplitude, so there is no intrinsic shape excitation. Low-lying surface phonons therefore begin at quadrupole order ($\lambda = 2$) [6, 1].

Bibliography

[1] R. A. Dunlap, *An Introduction to the Physics of Nuclei and Particles* (2nd ed.), IOP Publishing, Bristol (2023), DOI: 10.1088/978-0-7503-6094-4, Ch. 3 (Nuclear composition and size).

[2] W. Demtröder, *Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2023).

[3] S. Petrera, *Problems and Solutions in Nuclear and Particle Physics*, UNITEXT for Physics, Springer, Cham (2019).

[4] F. Terranova, *A Modern Primer in Particle and Nuclear Physics*, Oxford University Press (2021).

[5] S. D'Auria, *Introduction to Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2018).

[6] K. S. Krane, *Introductory Nuclear Physics*, Wiley, New York (1988).

[7] R. Hofstadter, "Electron scattering and nuclear structure," Rev. Mod. Phys. **28**, 214 (1956).

[8] H. F. Ehrenberg, R. Hofstadter, U. Meyer-Berkhout, D. G. Ravenhall, and S. E. Sobottka, "High-Energy Electron Scattering and the Charge Distribution of Carbon-12 and Oxygen-16," Phys. Rev. **113**, 666 (1959).

[9] H. Yukawa, "On the interaction of elementary particles. I," Proc. Phys.-Math. Soc. Japan **17**, 48 (1935).

[10] R. D. Woods and D. S. Saxon, "Diffuse surface optical model for nucleon-nuclei scattering," Phys. Rev. **95**, 577 (1954).

[11] H. G. J. Moseley, "The high-frequency spectra of the elements," Phil. Mag. **26**, 1024 (1913); **27**, 703 (1914).

[12] C. F. von Weizsäcker, "Zur Theorie der Kernmassen," Z. Phys. **96**, 431 (1935).

[13] P. Cappellaro, *Introduction to Applied Nuclear Physics* (MIT OpenCourseWare / LibreTexts), CC BY-NC-SA 4.0.

[14] J.-L. Basdevant, J. Rich, and M. Spiro, *Fundamentals in Nuclear Physics: From Nuclear Structure to Cosmology*, Springer, New York (2005).

[15] D. Tong, *Particle Physics*, Lecture notes (University of Cambridge / CERN school notes).

[16] M. G. Mayer, “On closed shells in nuclei. II,” Phys. Rev. **75**, 1969 (1949).

[17] O. Haxel, J. H. D. Jensen, and H. E. Suess, “On the “magic numbers” in nuclear structure,” Phys. Rev. **75**, 1766 (1949).

[18] T. Schmidt, “Über die magnetischen Momente der Atomkerne,” Z. Phys. **106**, 358 (1937).

[19] L. W. Nordheim, “On spins, moments, and shells in nuclei,” Phys. Rev. **78**, 294 (1950).

[20] Evaluated Nuclear Structure Data File (ENSDF), National Nuclear Data Center, Brookhaven National Laboratory, `https://www.nndc.bnl.gov/ensdf/`.

[21] NuDat 3, National Nuclear Data Center, Brookhaven National Laboratory, `https://www.nndc.bnl.gov/nudat3/`.

[22] N. J. Stone, “Table of nuclear magnetic dipole and electric quadrupole moments,” At. Data Nucl. Data Tables **90**, 75 (2005); updated compilations at NNDC.