Chapter 1

Schmidt Magnetic Moments

"In the extreme single-particle model the magnetic moment of an odd-A nucleus is attributed to the last unpaired nucleon."
standard shell-model statement

Lecture 8 treated the deuteron as a two-body magnetic moment. For heavier odd-A nuclei the extreme shell model makes a parallel claim: paired nucleons cancel, and μ is that of the valence nucleon in orbit $n\ell_j$. Plotting the resulting $\mu(j)$ gives the **Schmidt lines** [6, 1, 2].

1.1 Even–even nuclei: vanishing moment

In an even–even ground state every nucleon is paired to $I = 0$. Orbital and spin magnetism cancel within each pair, so

$$\mu = 0 \qquad (I^\pi = 0^+). \tag{1.1}$$

Experiment confirms that even–even ground states carry no static magnetic dipole (consistent with $I = 0$).

1.2 Nuclear magneton and g-factors

Magnetic moments of nuclei are measured in units of the nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p} \approx 5.05 \times 10^{-27}\,\mathrm{J\,T^{-1}}, \tag{1.2}$$

about 1/1836 of the Bohr magneton $\mu_B = e\hbar/(2m_e)$ because $m_p \gg m_e$. The z-component of the magnetic-moment operator for a single nucleon is

$$\mu_z = (g_\ell\,\ell_z + g_s\,s_z)\frac{\mu_N}{\hbar}, \tag{1.3}$$

with free-nucleon g-factors

$$\begin{aligned} g_\ell^{(p)} &= 1, \quad g_s^{(p)} = 5.5857, \\ g_\ell^{(n)} &= 0, \quad g_s^{(n)} = -3.8260. \end{aligned} \tag{1.4}$$

A useful way to remember the orbital factors is to begin with the magnetic moment of a particle of charge q and mass m:

$$\boldsymbol{\mu}_\ell = \frac{q}{2m}\,\boldsymbol{\ell} = \frac{q}{e}\frac{m_p}{m}\,\frac{\mu_N}{\hbar}\,\boldsymbol{\ell}. \tag{1.5}$$

For a proton, $q = e$ and $m \simeq m_p$, giving $g_\ell^{(p)} = 1$; for a neutron, $q = 0$, giving $g_\ell^{(n)} = 0$ in the free one-body operator. The spin factors follow from the measured intrinsic moments:

$$\mu_p = g_s^{(p)}\frac{\mu_N}{\hbar}\left(\frac{\hbar}{2}\right) = 2.7928\,\mu_N, \qquad \mu_n = \frac{g_s^{(n)}}{2}\mu_N = -1.9130\,\mu_N. \tag{1.6}$$

A structureless Dirac fermion would have $g_s = 2$. The large anomalous moments of the proton and neutron reflect their quark substructure: the proton (uud) and neutron (udd) carry internal charge currents even though the neutron is neutral overall. The neutron therefore has no orbital g_ℓ from net charge, yet a large spin moment [6, 5].

Because of the strong spin–orbit force, the good single-particle angular momentum is $\boldsymbol{j} = \boldsymbol{\ell}+\boldsymbol{s}$. The spectroscopic moment is the expectation value of μ_z in the stretched state $m_j = j$:

$$\mu = \langle j, m_j = j|\, g_\ell \ell_z + g_s s_z\, |j, m_j = j\rangle \frac{\mu_N}{\hbar}. \tag{1.7}$$

1.3 Case $j = \ell + \frac{1}{2}$: unique m_ℓ, m_s

For $j = \ell + \frac{1}{2}$ and $m_j = j = \ell + \frac{1}{2}$, the only combination consistent with $m_\ell + m_s = m_j$ is

$$m_\ell = \ell, \qquad m_s = +\tfrac{1}{2}. \tag{1.8}$$

The coupled state is therefore a single uncoupled basis vector,

$$|j = \ell + \tfrac{1}{2}, m_j = j\rangle = |\ell, m_\ell = \ell\rangle|\tfrac{1}{2}, m_s = \tfrac{1}{2}\rangle. \tag{1.9}$$

Consequently

$$\langle \ell_z\rangle = \ell\hbar, \qquad\qquad \langle s_z\rangle = \tfrac{1}{2}\hbar, \tag{1.10}$$

$$\begin{aligned}\frac{\mu}{\mu_N} &= \frac{1}{\hbar}\left(g_\ell \ell\hbar + g_s\frac{\hbar}{2}\right)\\ &= g_\ell \ell + \tfrac{1}{2}g_s.\end{aligned} \tag{1.11}$$

Using $\ell = j - \frac{1}{2}$ gives, one algebraic step at a time,

$$\frac{\mu}{\mu_N} = g_\ell\left(j - \tfrac{1}{2}\right) + \tfrac{1}{2}g_s = j\,g_\ell + \tfrac{1}{2}(g_s - g_\ell). \tag{1.12}$$

Specialising to free g-factors gives the first Schmidt formula

$$\frac{\mu}{\mu_N} = \begin{cases} j + 2.293 & \text{odd proton } (g_\ell = 1,\ g_s = 5.586),\\ -1.913 & \text{odd neutron } (g_\ell = 0,\ g_s = -3.826).\end{cases} \tag{1.13}$$

For a $d_{5/2}$ proton ($\ell = 2$, $j = \frac{5}{2}$), $\mu \approx 4.79\,\mu_N$ on the Schmidt line. For any odd neutron with $j = \ell + \frac{1}{2}$, the moment is simply half the free-neutron moment (independent of j), because only spin contributes.

1.4 Case $j = \ell - \frac{1}{2}$: two mixed components

For $j = \ell - \frac{1}{2}$ and $m_j = j$, two (m_ℓ, m_s) pairs contribute:

$$\begin{aligned} |1\rangle &= |m_\ell = \ell,\ m_s = -\tfrac{1}{2}\rangle, \\ |2\rangle &= |m_\ell = \ell - 1,\ m_s = +\tfrac{1}{2}\rangle. \end{aligned} \tag{1.14}$$

The stretched state is the normalised combination

$$|j, m_j = j\rangle = a|1\rangle + b|2\rangle. \tag{1.15}$$

1.4.1 Coefficients from $J_+|j,j\rangle = 0$

The raising operator $J_+ = L_+ + S_+$ annihilates the highest-weight state. The two ladder operators act according to

$$L_+|\ell, m_\ell\rangle = \hbar\sqrt{\ell(\ell+1) - m_\ell(m_\ell+1)}|\ell, m_\ell + 1\rangle, \tag{1.16}$$

$$S_+|\tfrac{1}{2}, m_s\rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - m_s(m_s+1)}|\tfrac{1}{2}, m_s + 1\rangle. \tag{1.17}$$

Therefore

$$J_+|1\rangle = (L_+ + S_+)|\ell, -\tfrac{1}{2}\rangle = \hbar|\ell, +\tfrac{1}{2}\rangle, \tag{1.18}$$

$$J_+|2\rangle = (L_+ + S_+)|\ell - 1, +\tfrac{1}{2}\rangle = \hbar\sqrt{2\ell}\,|\ell, +\tfrac{1}{2}\rangle. \tag{1.19}$$

Here the compact ket on the right denotes $|m_\ell = \ell, m_s = +\frac{1}{2}\rangle$. Acting on the mixture now gives

$$\begin{aligned} 0 = J_+|j,j\rangle &= aJ_+|1\rangle + bJ_+|2\rangle \\ &= \hbar(a + b\sqrt{2\ell})|\ell, +\tfrac{1}{2}\rangle. \end{aligned} \tag{1.20}$$

Since the ket and $\hbar$ are nonzero, the coefficient must vanish:

$$a + b\sqrt{2\ell} = 0 \qquad \Rightarrow \qquad a = -b\sqrt{2\ell}, \tag{1.21}$$

together with normalisation $|a|^2 + |b|^2 = 1$. Therefore

$$|a|^2 = \frac{2\ell}{2\ell+1}, \qquad |b|^2 = \frac{1}{2\ell+1}. \tag{1.22}$$

Choosing an irrelevant overall phase so that $b > 0$, the normalized state can be written explicitly as

$$|j = \ell - \tfrac{1}{2}, j\rangle = -\sqrt{\frac{2\ell}{2\ell+1}}\,|1\rangle + \sqrt{\frac{1}{2\ell+1}}\,|2\rangle. \tag{1.23}$$

These are precisely the Clebsch–Gordan coefficients for coupling ℓ and $s = \frac{1}{2}$ to $j = \ell - \frac{1}{2}$.

1.4.2 Expectation value of μ_z

Diagonal matrix elements of the magnetic operator are

$$\begin{aligned} \langle 1|\mu_z/\mu_N|1\rangle &= g_\ell \ell - \tfrac{1}{2} g_s, \\ \langle 2|\mu_z/\mu_N|2\rangle &= g_\ell(\ell - 1) + \tfrac{1}{2} g_s, \end{aligned} \tag{1.24}$$

and cross terms vanish by orthogonality of the $|m_\ell m_s\rangle$ basis. More explicitly, ℓ_z and s_z are diagonal in this basis, so

$$\langle 1|\ell_z|2\rangle = \langle 1|s_z|2\rangle = 0. \tag{1.25}$$

This diagonality, not orthogonality alone for an arbitrary operator, is the reason the interference terms vanish. Expanding the full matrix element,

$$\begin{aligned} \frac{\mu}{\mu_N} &= (a^*\langle 1| + b^*\langle 2|) \frac{g_\ell \ell_z + g_s s_z}{\hbar} (a|1\rangle + b|2\rangle) \\ &= |a|^2 \langle 1|\mu_z/\mu_N|1\rangle + |b|^2 \langle 2|\mu_z/\mu_N|2\rangle \\ &\quad + a^* b \langle 1|\mu_z/\mu_N|2\rangle + b^* a \langle 2|\mu_z/\mu_N|1\rangle. \end{aligned} \tag{1.26}$$

The last line is zero, and the weighted average is therefore

$$\frac{\mu}{\mu_N} = |a|^2 (g_\ell \ell - \tfrac{1}{2} g_s) + |b|^2 (g_\ell(\ell - 1) + \tfrac{1}{2} g_s). \tag{1.27}$$

Insert $|a|^2 = 2\ell/(2\ell + 1)$ and $|b|^2 = 1/(2\ell + 1)$:

$$\begin{aligned} \frac{\mu}{\mu_N} &= \frac{2\ell}{2\ell + 1}(g_\ell \ell - \tfrac{1}{2} g_s) + \frac{1}{2\ell + 1}(g_\ell(\ell - 1) + \tfrac{1}{2} g_s) \\ &= \frac{1}{2\ell + 1}\Big[2\ell^2 g_\ell - \ell g_s + (\ell - 1) g_\ell + \tfrac{1}{2} g_s\Big] \\ &= \frac{1}{2\ell + 1}\Big[(2\ell^2 + \ell - 1) g_\ell - (\ell - \tfrac{1}{2}) g_s\Big]. \end{aligned} \tag{1.28}$$

Now set $j = \ell - \frac{1}{2}$, so $\ell = j + \frac{1}{2}$ and $2\ell + 1 = 2j + 2$. The two coefficients in the numerator become

$$2\ell^2 + \ell - 1 = 2(j + \tfrac{1}{2})^2 + (j + \tfrac{1}{2}) - 1 = 2j^2 + 3j = j(2j + 3), \tag{1.29}$$

$$\ell - \tfrac{1}{2} = j. \tag{1.30}$$

Thus

$$\begin{aligned} \frac{\mu}{\mu_N} &= \frac{j(2j + 3) g_\ell - j g_s}{2(j + 1)} \\ &= \frac{j}{2(j + 1)}\big[(2j + 3) g_\ell - g_s\big]. \end{aligned} \tag{1.31}$$

Adding and subtracting $jg_\ell/[2(j+1)]$ in the last expression gives the equivalent standard Schmidt form

$$\frac{\mu}{\mu_N} = \frac{j}{2(j+1)}\left[(2j+3)g_\ell - g_s\right] = j\,g_\ell - \frac{j}{2(j+1)}(g_s - g_\ell). \tag{1.32}$$

Specialising to free g-factors:

$$\frac{\mu}{\mu_N} = \begin{cases} \dfrac{j}{j+1}\,(j - \tfrac{1}{2}g_s^{(p)} + \tfrac{3}{2}) = \dfrac{j(j-1.293)}{j+1} & \text{odd proton,} \\ -\dfrac{j}{2(j+1)}\,g_s^{(n)} = +1.913\,\dfrac{j}{j+1} & \text{odd neutron.} \end{cases} \tag{1.33}$$

For an odd neutron in $p_{1/2}$ ($j = \frac{1}{2}$), $\mu/\mu_N = 1.913 \times \frac{1}{2}/\frac{3}{2} = 0.638$. These limiting curves are the Schmidt lines first discussed by Schmidt in 1937 [18, 6, 1].

1.4.3 Independent check: the projection theorem

The same result follows without writing Clebsch–Gordan coefficients. Rotational symmetry requires the expectation value of any vector operator in $|jm_j\rangle$ to point along $\boldsymbol{j}$. Hence

$$\langle \ell_z \rangle = \frac{\langle \boldsymbol{\ell}\cdot\boldsymbol{j}\rangle}{j(j+1)\hbar^2}\,m_j\hbar, \qquad \langle s_z \rangle = \frac{\langle \boldsymbol{s}\cdot\boldsymbol{j}\rangle}{j(j+1)\hbar^2}\,m_j\hbar. \tag{1.34}$$

Using $\boldsymbol{j} = \boldsymbol{\ell} + \boldsymbol{s}$,

$$2\boldsymbol{\ell}\cdot\boldsymbol{j} = \boldsymbol{j}^2 + \boldsymbol{\ell}^2 - \boldsymbol{s}^2, \tag{1.35}$$

$$2\boldsymbol{s}\cdot\boldsymbol{j} = \boldsymbol{j}^2 - \boldsymbol{\ell}^2 + \boldsymbol{s}^2. \tag{1.36}$$

For the spectroscopic state $m_j = j$, division by $\hbar$ gives

$$\frac{\langle \ell_z \rangle}{\hbar} = \frac{j(j+1) + \ell(\ell+1) - s(s+1)}{2(j+1)}, \tag{1.37}$$

$$\frac{\langle s_z \rangle}{\hbar} = \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2(j+1)}. \tag{1.38}$$

Substitution into $\mu/\mu_N = g_\ell\langle \ell_z\rangle/\hbar + g_s\langle s_z\rangle/\hbar$, with $s = \frac{1}{2}$, reproduces Eqs. (1.13) and (1.32). This is the same projection theorem used for the deuteron in Lecture 8; it is a valuable check because it keeps every orbital and spin term visible.

1.4.4 Why a one-hole state has the same Schmidt value

A filled j subshell has zero total moment because the contributions from m and $-m$ cancel. To construct a hole state with total $M = +j$, remove the particle with $m = -j$. The remaining shell therefore has

$$\mu_{\text{hole}}(M = +j) = 0 - \mu_{\text{particle}}(m = -j) = \mu_{\text{particle}}(m = +j), \tag{1.39}$$

because a vector moment changes sign under $m \to -m$. Thus a particle and a hole in the same j orbit have the same Schmidt moment in the extreme model, even though their many-body occupations are complementary.

1.5 Schmidt lines and experiment

Plotting μ against j for odd-proton nuclei gives two rising lines ($j = \ell \pm \frac{1}{2}$). For odd-neutron nuclei the $j = \ell + \frac{1}{2}$ line is flat at $-1.913\,\mu_N$ and the $j = \ell - \frac{1}{2}$ line is positive and slowly rising. Experimental ground-state moments of odd-A nuclei typically fall *between* the two Schmidt lines for the assigned j (Fig. 1.1) [18, 6, 2, 22]. The Schmidt curves are theoretical single-particle limits, not rigorous mathematical bounds; they organise the data and diagnose configuration purity.

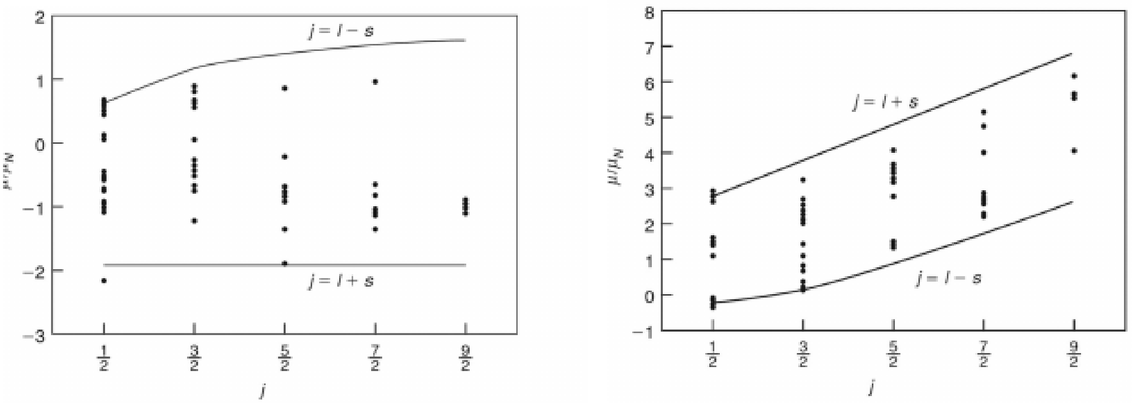

Figure 1.1: Schmidt lines and data: left, odd-N (neutron) moments; right, odd-Z (proton) moments (re-extracted from the full-resolution original PH5001 lecture source page; cf. Dunlap [1], Krane [6], and Stone [22]).

1.6 Why real nuclei lie between the Schmidt lines

The extreme single-particle picture is too simple:

- configuration mixing admixes other orbitals with the same I^π;
- meson-exchange currents modify the free one-body magnetic operator inside the nucleus;
- collective core polarisation contributes for soft nuclei far from closed shells;
- effective spin g-factors are often reduced from free values. A common phenomenological estimate is $g_s^{\text{eff}} \approx 0.7\, g_s^{\text{free}}$, but this is not a universal law and should be fitted orbit by orbit [6, 1].

Near doubly magic cores (e.g. ^{17}O, ^{41}Ca) agreement with Schmidt values is better; mid-shell nuclei deviate more [6, 2, 1, 22]. When a measured μ lies close to one Schmidt line, the valence assignment $j = \ell \pm \frac{1}{2}$ is strongly supported; intermediate values tag mixing.

Key idea

Schmidt lines are the shell-model baseline for μ. They fix the single-particle g-factors and $j = \ell \pm \frac{1}{2}$ assignment; deviations measure correlations beyond the extreme model.

1.7 Further physics: magnetic moments as configuration tags

Measured magnetic moments are among the most sensitive tests of the valence configuration near closed shells. If μ lies close to one Schmidt line, the extreme single-particle assignment $j = \ell \pm \frac{1}{2}$ is strongly supported. Intermediate values signal configuration mixing or collective admixtures [6, 2, 22].

Effective g_s factors reduced from the free nucleon values are a compact way to summarise medium modifications, but they absorb several distinct mechanisms: core polarisation, meson-exchange currents, and truncated model spaces. For that reason one should not treat a single quenching factor as fundamental. Near doubly magic cores such as ^{16}O and ^{40}Ca, moments of neighbouring odd-A nuclei remain comparatively close to Schmidt predictions; far from magic numbers, collective contributions dominate and the Schmidt picture is only a reference baseline.

1.8 Problems with solutions

Problem 14.1 — Schmidt value for $d_{5/2}$ proton

Compute the Schmidt moment for an odd proton in $j = \ell + \frac{1}{2} = \frac{5}{2}$.

Solution 14.1

For $j = \ell + \frac{1}{2}$ and free proton g-factors,

$$\frac{\mu}{\mu_N} = j\, g_\ell + \tfrac{1}{2}(g_s - g_\ell) = \tfrac{5}{2} \cdot 1 + \tfrac{1}{2}(5.586 - 1) = 2.5 + 2.293 = 4.793.$$

Equivalently, $\mu/\mu_N = j + 2.293$ [18, 6].

Problem 14.2 — Neutron $p_{1/2}$ Schmidt moment

For an odd neutron with $j = \ell - \frac{1}{2} = \frac{1}{2}$ ($\ell = 1$), compute the Schmidt moment in nuclear magnetons.

Solution 14.2

For an odd neutron with $j = \ell - \frac{1}{2}$,

$$\frac{\mu}{\mu_N} = +1.913\, \frac{j}{j+1} = 1.913 \times \frac{1/2}{3/2} = \frac{1.913}{3} \approx 0.638.$$

Equivalently, $\mu/\mu_N = -g_s^{(n)} j/(2(j+1))$ with $g_s^{(n)} = -3.826$ yields the same result [6, 18].

Problem 14.3 — Neutron $j = \ell + \frac{1}{2}$

Show that any odd neutron with $j = \ell + \frac{1}{2}$ has Schmidt moment $\mu = -1.913\, \mu_N$, independent of j.

Solution 14.3

With $g_\ell^{(n)} = 0$ and $g_s^{(n)} = -3.826$, the $j = \ell + \frac{1}{2}$ formula collapses to

$$\frac{\mu}{\mu_N} = \tfrac{1}{2} g_s = -1.913,$$

for every such orbit. Only the spin moment contributes [6, 1].

Problem 14.4 — Coefficients a and b

For $j = \ell - \frac{1}{2}$ and $m_j = j$, derive $|a|^2 = 2\ell/(2\ell+1)$ and $|b|^2 = 1/(2\ell+1)$ from $J_+|j, j\rangle = 0$.

Solution 14.4

Write $|j, j\rangle = a|1\rangle + b|2\rangle$ with $|1\rangle = |\ell, -\frac{1}{2}\rangle$ and $|2\rangle = |\ell - 1, +\frac{1}{2}\rangle$. Acting with $J_+ = L_+ + S_+$ and using $L_+|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|\ell, m+1\rangle$ gives the condition $a + b\sqrt{2\ell} = 0$. Together with $|a|^2 + |b|^2 = 1$ one obtains

$$|a|^2 = \frac{2\ell}{2\ell+1}, \qquad |b|^2 = \frac{1}{2\ell+1},$$

as in the main text [6].

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