Chapter 1

Quadrupole Moment and Tensor Force

"The mixing of ℓ values in the deuteron is the best evidence we have for the noncentral (tensor) character of the nuclear force."

K. S. Krane

1.1 Electric quadrupole moment from the multipole expansion

The magnetic moment of the deuteron is close to, but not equal to, the sum of the free nucleon moments. That small discrepancy already suggests a few percent D-wave admixture. A sharper geometric probe is the electric quadrupole moment: it reports the *shape* of the charge distribution. The definition used in nuclear physics follows from the multipole expansion of the electrostatic potential of a localised charge density, exactly as in the original lecture (Fig. 1 below) [6, 1, 2].

1.1.1 Potential of a nuclear charge distribution (Fig. 1)

Let $\rho(\boldsymbol{r}')$ be the charge density of the nucleus. The electric potential at an exterior point $\boldsymbol{r}$ is

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, \mathrm{d}^3 r', \tag{1.1}$$

where $\boldsymbol{r}'$ labels a charge element inside the nucleus (Figure 1.1).

1.1.2 Multipole expansion far from the nucleus

Far from the nucleus one expands the Coulomb kernel in Legendre polynomials. Write

$$|\boldsymbol{r}-\boldsymbol{r}'| = r\left(1 - 2\frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2\right)^{1/2},$$

with θ the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. For $r > r'$ the generating function of Legendre polynomials is

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta). \tag{1.2}$$

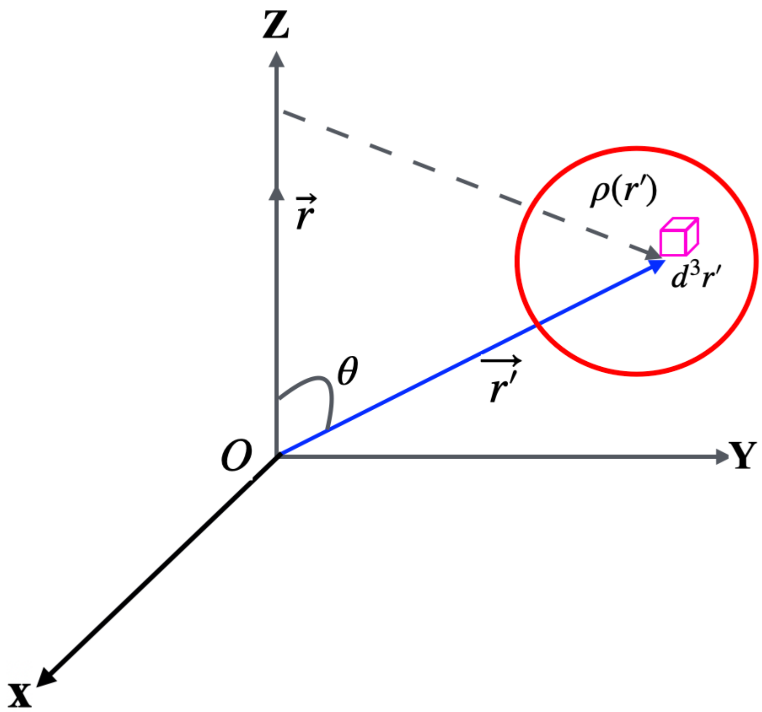

Figure 1.1: Charge distribution $\rho(\boldsymbol{r}')$ inside a nucleus (Fig. 1 of the lecture). An infinitesimal charge element occupies volume d^3r' at $\boldsymbol{r}'$. The field point $\boldsymbol{r}$ lies outside the distribution; θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$.

The first three polynomials are $P_0 = 1$, $P_1(\cos\theta) = \cos\theta$, and $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$, so

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \frac{r'^2}{2r^2}(3\cos^2\theta - 1) + \cdots\right). \tag{1.3}$$

The quadrupole term is precisely the $\ell = 2$ piece of this expansion: it is of order $(r'/r)^2$ and must be retained if one wishes to discuss Q. (Omitting r'^2/r^2 would truncate before the quadrupole appears.)

Substitute (1.3) into (1.1):

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{r}')\frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \frac{r'^2}{2r^2}(3\cos^2\theta - 1) + \cdots\right)\mathrm{d}^3r'. \tag{1.4}$$

Collecting powers of $1/r$ and writing the integrated moments in standard form,

$$\Phi(\boldsymbol{r}) = \frac{e}{4\pi\varepsilon_0 r}\left(Z\,P_0 + \frac{D}{r}\,P_1(\cos\theta) + \frac{Q}{r^2}\,P_2(\cos\theta) + \cdots\right), \tag{1.5}$$

where

- the **monopole** Z is the total charge (in units of e);
- D is the **electric dipole** moment;
- Q is the **electric quadrupole** moment.

Nuclear ground states of definite parity have vanishing static electric dipole. The first multipole that can report a permanent deformation is therefore the quadrupole.

1.1.3 Definition of the quadrupole tensor

The Cartesian quadrupole tensor associated with the third term of the expansion is

$$Q_{ij} = \int \rho(\boldsymbol{r})(3x_i x_j - r^2\delta_{ij})\,\mathrm{d}^3 r, \tag{1.6}$$

with $i, j = x, y, z$ and δ_{ij} the Kronecker symbol. Examples:

$$Q_{xx} = \int \rho(\boldsymbol{r})\,(3x^2 - r^2)\,\mathrm{d}^3 r, \tag{1.7}$$

$$Q_{xy} = \int \rho(\boldsymbol{r})\,(3xy)\,\mathrm{d}^3 r. \tag{1.8}$$

There are nine components, but only five are independent (the tensor is symmetric and traceless). In nuclear spectroscopy the component measured in the stretched state is almost always

$$Q \equiv Q_{zz} = \int \rho(\boldsymbol{r})\,(3z^2 - r^2)\,\mathrm{d}^3 r = e\langle 3z^2 - r^2\rangle \tag{1.9}$$

(when ρ is written so that eQ has the dimension of charge times area; tables often quote Q in barns). Classically, $Q > 0$ for a prolate (cigar-like) distribution along z, and $Q < 0$ for an oblate (flattened) one.

1.1.4 Spherical symmetry forces vanishing quadrupole

If the charge density is spherically symmetric, $\rho = \rho(r)$ only, then by symmetry

$$\int z^2\rho\,\mathrm{d}^3 r = \int x^2\rho\,\mathrm{d}^3 r = \int y^2\rho\,\mathrm{d}^3 r = \frac{1}{3}\int r^2\rho\,\mathrm{d}^3 r. \tag{1.10}$$

Hence $3z^2 - r^2$ averages to zero and

$$Q_{zz} = \int \rho(\boldsymbol{r}')\,(3z^2 - r^2)\,\mathrm{d}^3 r' = 0. \tag{1.11}$$

Equivalently, on a sphere $x^2 = y^2 = z^2$ so $r^2 = 3z^2$ and $3z^2 - r^2 = 0$ pointwise in the average sense of (1.11). **A spherical charge distribution has no electric quadrupole moment.**

1.2 Deuteron quadrupole moment

1.2.1 Pure S-wave implies vanishing Q_d

Free protons and neutrons have vanishing electric quadrupole moments. Any nonzero Q of the deuteron must come from the *orbital* distribution of the proton's charge about the centre of mass.

If the NN potential were purely central, the ground state would be pure $L = 0$. The spatial wave function is then proportional to the spherical harmonic Y_{00}, which is angle-independent. The charge density is therefore spherically symmetric, and the multipole argument of Section 1.1.4

applies at once:

$$Q_d(L{=}0) = 0. \tag{1.12}$$

A pure s-wave deuteron would look electromagnetically spherical.

1.2.2 The measured value

Experiment finds a small but clearly nonzero moment for the bound deuteron [6]:

$$Q_d = Q_{zz} = 2.88 \times 10^{-27}\,\mathrm{cm}^2 = 0.00288\,\mathrm{b} = 0.288\,\mathrm{e}\cdot\mathrm{fm}^2. \tag{1.13}$$

(The unit is area, as required by (1.9); one barn $= 10^{-28}\,\mathrm{m}^2 = 100\,\mathrm{fm}^2$.) The value is tiny compared with collective rare-earth rotors (several barns), but it is not zero. That single experimental fact rules out a pure $L = 0$ ground state and, with it, a purely central NN potential.

A geometric scale eR^2 with $R \sim 2\,\mathrm{fm}$ is a few $e\cdot\mathrm{fm}^2$; the measured $Q_d \sim 0.3\,e\cdot\mathrm{fm}^2$ is smaller, consistent with a mild deformation from a few-percent D-wave rather than a strongly deformed rotor.

1.2.3 Mixed wave function and the D-state

Using the mixed ground state of Lecture 8,

$$|\psi\rangle = a_S|S\rangle + a_D|D\rangle, \qquad |a_S|^2 + |a_D|^2 = 1,$$

the spectroscopic quadrupole moment in the stretched state is

$$Q_d = \langle\psi|\hat{Q}_{zz}|\psi\rangle = |a_S|^2 Q_{SS} + 2\,\mathrm{Re}(a_S^* a_D)\,Q_{SD} + |a_D|^2 Q_{DD}. \tag{1.14}$$

The pure-S matrix element vanishes by spherical symmetry, $Q_{SS} = 0$. The interference and pure-D pieces may be written in terms of the radial functions $u(r) = rR_S(r)$ and $w(r) = rR_D(r)$ (standard Blatt–Weisskopf / Reid normalisation) as

$$Q_d = \frac{\sqrt{2}}{10}\int_0^\infty r^2\,\big(uw - \tfrac{w^2}{\sqrt{8}}\big)\mathrm{d}r \quad \text{(in units of } e), \tag{1.15}$$

up to an overall conventional factor of e when Q is quoted in $e\cdot\mathrm{fm}^2$. Equivalently, the dominant contribution is the S–D interference term $\propto a_S a_D$; the pure D–D piece is smaller because P_D itself is small. Evaluating the radial integrals requires a model for $u(r)$ and $w(r)$. Realistic potentials fitted to NN scattering reproduce $Q_d \approx 0.286\,e\cdot\mathrm{fm}^2$ with a few-percent D-state probability, consistent with the magnetic-moment estimate of Lecture 8 once exchange currents are acknowledged [6]. The positive sign of Q_d means a prolate (cigar-like) charge distribution along the spin axis.

Key idea

Two multipoles, one structure.

Observable	Pure S prediction	Experiment
μ_d/μ_N	0.880	0.857
Q_d	0	0.00288 b

Both point to a small D-wave. The quadrupole moment is an on/off signal of nonsphericity; the magnetic moment is a percent-level correction to an already large spin moment.

Caution

Extracting P_D from Q_d alone is model dependent because the D-state radial wave function is not measured directly. Potentials that produce a few-percent D-state also produce the observed Q_d, and scattering analyses agree. The qualitative need for a tensor force does not rest on a single fitted percentage [6].

1.3 Non-central force and the tensor operator

1.3.1 Why a central potential cannot mix L

In a purely central potential $V(r)$, orbital angular momentum is a constant of the motion: $[\boldsymbol{L}^2, H] = 0$. Energy eigenstates may therefore be chosen with definite L. A single eigenstate cannot be a coherent superposition of $L = 0$ and $L = 2$. Experiment, however, requires precisely that mixture: $Q_d \neq 0$ and the magnetic-moment deficit. The residual NN interaction therefore cannot be purely central; it must contain a piece that does not commute with $\boldsymbol{L}^2$.

Spin–spin forces of the form $V_\sigma(r)\,\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ still depend only on the relative distance: they split triplet from singlet but do not mix different L. The operator that couples S and D must involve the orientation of the relative vector $\boldsymbol{r}$ with respect to the nucleon spins.

1.3.2 Magnetic-dipole analogy and the tensor operator

A familiar non-central interaction is the force between two magnetic dipoles. At fixed separation the energy depends on how the moments are oriented relative to $\hat{\boldsymbol{r}}$: collinear and sideways arrangements are not equivalent.

Classically,

$$U \propto \frac{3(\boldsymbol{m}_1 \cdot \hat{r})(\boldsymbol{m}_2 \cdot \hat{r}) - \boldsymbol{m}_1 \cdot \boldsymbol{m}_2}{r^3}.$$

Replacing magnetic moments by nucleon spin operators yields the **tensor operator**

$$S_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{r})(\boldsymbol{\sigma}_2 \cdot \hat{r}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \tag{1.16}$$

and a potential piece $V_T(r)\,S_{12}$. In the coupled ${}^3S_1$–${}^3D_1$ subspace the angular matrix elements of S_{12} are standard:

$$\langle {}^3S_1 | S_{12} | {}^3S_1 \rangle = 0, \quad \langle {}^3S_1 | S_{12} | {}^3D_1 \rangle = \sqrt{8}, \quad \langle {}^3D_1 | S_{12} | {}^3D_1 \rangle = -2. \tag{1.17}$$

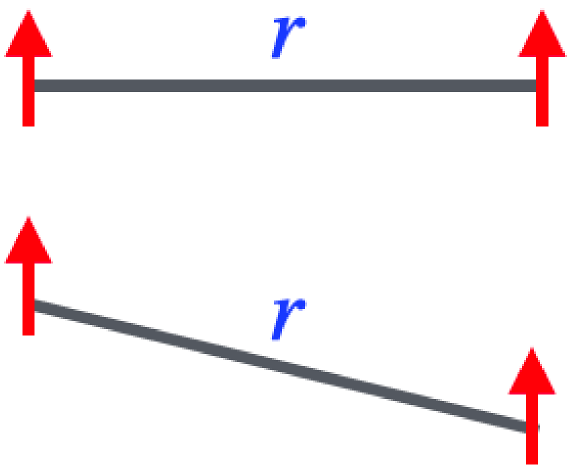

Figure 1.2: Two dipoles at fixed r: collinear versus tilted. The force depends on orientation, so it is non-central.

(The vanishing S–S diagonal shows that a pure central spin–spin force cannot mix L; only the off-diagonal tensor matrix element does.) Writing the radial wave as $\psi = u(r)Y_S + w(r)Y_D$, the coupled radial Schrödinger equations in the deuteron channel become

$$-\frac{\hbar^2}{m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + V_C u + \sqrt{8}\,V_T w = Eu, \tag{1.18}$$

$$-\frac{\hbar^2}{m}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} - \frac{6w}{r^2}\right) + (V_C - 2V_T - 3V_{LS})w + \sqrt{8}\,V_T u = Ew, \tag{1.19}$$

where V_C collects central (including spin–spin) pieces and V_{LS} is the spin–orbit term. The off-diagonal $\sqrt{8}\,V_T$ is precisely what mixes u and w and generates $Q_d \neq 0$.

One-pion exchange produces exactly this tensor structure at long range, with radial factors $e^{-m_\pi r}/r$ and polynomials in $1/(m_\pi r)$ [6, 2, 4, 9]. In the deuteron channel V_T mixes $^3S_1$ with $^3D_1$, generates $Q_d \neq 0$, and shifts μ_d by a few percent. That mixing is the clearest bound-state evidence for the tensor character of the residual nuclear force [6].

Definition

Tensor force. The non-central NN potential contains $V_T(r)S_{12}$. In the deuteron it is responsible for S–D mixing, for $Q_d \neq 0$, and for the small correction to μ_d.

1.3.3 The full NN potential is richer

Phenomenological potentials include central, spin–spin, tensor, spin–orbit, and short-range terms, with isospin dependence. The deuteron bound state has taught us the first three. Scattering (all partial waves, both singlet and triplet, as functions of energy) fixes the rest. When the bound state runs out of information, the continuum (scattering) takes over.

1.4 Quadrupole moments of heavier nuclei (context)

Tables of quadrupole moments place the deuteron in perspective. Many nuclei have $|Q|$ of order eR^2 (single-particle scale). Rare-earth nuclei reach several barns: the whole proton distribution is collectively deformed. Closed-shell nuclei have $Q \approx 0$. For deformed heavy nuclei one distinguishes the **intrinsic** quadrupole moment Q_0 in the body-fixed frame from the **spectroscopic** moment measured in the laboratory; they differ by a projection factor depending on I. The deuteron is the few-body extreme: weak binding, small spectroscopic Q, tensor-driven rather than collective [6, 1].

1.5 Synthesis: Lectures 7–9

Lecture	Observable	Lesson
7	$B_d \approx 2.224\,\mathrm{MeV}$	Range $\sim 2\,\mathrm{fm}$, depth $\sim 35\,\mathrm{MeV}$; barely bound
8	$I^\pi = 1^+$, $\mu_d = 0.857\,\mu_N$	Triplet; $\sim 96\%$ S + $\sim 4\%$ D; spin-dependent force
9	$Q_d = 0.00288\,\mathrm{b}$	Nonspherical; tensor force $V_T S_{12}$

Takeaway

Physics of Lecture 9.

- $Q = \langle 3z^2 - r^2 \rangle$: vanishes for spherical charge distributions.
- $Q_d = 0.00288\,\mathrm{b} \neq 0$: the deuteron is not a pure S-state.
- Consistent few-percent D-wave from μ_d and Q_d.
- Tensor operator S_{12} encodes the non-central force; pion exchange is its long-range origin.
- Lectures 7–9 define the qualitative NN force; Lectures 10–11 develop scattering phase shifts and meson exchange for the course.

1.6 Deeper physics: spectroscopic Q, shells, and one-pion tensor force

1.6.1 Intrinsic vs spectroscopic quadrupole moment

For deformed heavy nuclei one distinguishes the **intrinsic** quadrupole moment Q_0 in the body-fixed frame from the **spectroscopic** moment Q measured in the lab. They are related by a Clebsch–Gordan projection factor depending on I. For the deuteron $I = 1$ there is no collective rotor picture: Q_d is a few-body matrix element of $r^2 Y_{20}$ between S and D components. Standard values are

$$Q_d \approx +0.286\,\mathrm{e} \cdot \mathrm{fm}^2 \tag{1.20}$$

and stresses that free nucleons have $Q = 0$, so any nonzero deuteron Q is purely orbital / deformation physics [6].

1.6.2 Shell-model systematics of Q

The plot of quadrupole moments versus Z shows sign changes near magic numbers: closed shells are spherical ($Q \approx 0$); a valence proton hole just below a shell tends to give prolate $Q > 0$, while a particle just above tends toward oblate $Q < 0$ in the single-particle picture. Collective mid-shell nuclei have Q many times the single-particle estimate. The deuteron sits at the opposite extreme: no shells, weak binding, small Q generated by tensor-driven D-wave admixture [1].

1.6.3 One-pion exchange and S_{12}

The long-range NN force is one-pion exchange (Yukawa). The pseudoscalar pion coupling produces a potential containing the tensor operator S_{12} with radial dependence

$$V_T(r) \propto \frac{e^{-m_\pi r}}{r}\left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right) \tag{1.21}$$

(up to spin–isospin factors). The angular matrix elements (1.17) then feed into the coupled equations (1.18)–(1.19), so pion exchange is the microscopic origin of the magnetic-dipole analogy in the lecture: it is non-central and mixes ${}^3S_1$–${}^3D_1$. Short-range repulsion, spin–orbit forces, and multi-pion / contact terms complete modern potentials (Argonne, CD-Bonn, chiral EFT). The deuteron Q_d and η_D (D/S asymptotic ratio) are primary constraints on V_T [6, 4, 2].

Key idea

Tensor force checklist.

- Operator: $S_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{r})(\boldsymbol{\sigma}_2 \cdot \hat{r}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$.
- Bound-state signals: $Q_d \neq 0$, $\mu_d \neq \mu_p + \mu_n$, few-percent D-state probability (model dependent).
- Continuum signals: mixing parameters in NN phase-shift analyses (ε_1, etc.).
- Microscopic origin: one-pion exchange at long range.

1.6.4 What remains for scattering

Bound-state multipoles fix the deuteron channel. Scattering states access all partial waves, both singlet and triplet, as functions of energy. Phase shifts $\delta_{{}^1S_0}(E)$, $\delta_{{}^3S_1}(E)$, P- and D-wave phase shifts, and mixing angles complete the NN interaction and feed into many-body nuclear structure. That programme is the natural continuation of Lectures 7–9 [2, 6].

1.7 Further physics: multipoles beyond the quadrupole

Outside a localised charge distribution the electrostatic potential admits a multipole expansion: monopole (total charge), dipole (vanishes for definite-parity ground states), quadrupole, octupole, ... At large distance the lowest non-vanishing multipole dominates [13, 1, 6].

The same logic applies to magnetic multipoles and to radiation. In γ decay, multipole selection rules (angular momentum and parity of the photon field) organise transition rates (a topic for later lectures). For the deuteron ground state the essential static multipoles are already in hand: magnetic dipole μ_d and electric quadrupole Q_d. Together they require a tensor force and a few-percent D-wave, completing the qualitative NN picture of Lectures 7–9.

1.7.1 Deformation and quadrupole moments of heavier nuclei

Nuclei near closed (magic) shells are nearly spherical and have small Q. Nuclei far from magic numbers are often permanently deformed and can have large spectroscopic quadrupole moments; rotational bands appear in the spectrum [14, 6, 13]. The deuteron is the opposite extreme: few-body, weakly deformed, with Q_d generated by a small D-wave rather than collective rotation.

Static multipoles (monopole, dipole, quadrupole, ...) also organise electromagnetic radiation: γ transitions carry multipolarity $E1, M1, E2, \ldots$ with selection rules fixed by angular momentum and parity [13], the dynamical counterpart of the static moments used here.

1.7.2 Deuteron quadrupole moment in numbers

The free neutron and proton have vanishing electric quadrupole moments, so any $Q_d \neq 0$ must come from orbital structure. Experiment gives

$$Q_d = +0.00288\,\mathrm{b} = +0.286\,\mathrm{fm}^2, \tag{1.22}$$

small compared with heavy deformed nuclei but definitely nonzero [6]. A pure ${}^3S_1$ wave function would yield $Q_d = 0$. A mixed $S{+}D$ wave function produces

$$Q_d \propto a_S a_D \,\langle r^2\rangle_{SD} + a_D^2 \,\langle r^2\rangle_{DD}, \tag{1.23}$$

so Q_d is linearly sensitive to the S–D interference. Realistic potentials that fit scattering and B_d reproduce Q_d with a_D^2 of a few percent, consistent with the magnetic-moment estimate [6, 14].

The positive sign of Q_d corresponds to a prolate (cigar-like) deformation: more charge along the symmetry axis than in the equatorial plane.

1.7.3 Single-particle vs collective quadrupole moments

For a single valence proton in an orbit of mean-square radius $\langle r^2\rangle \sim R^2 \sim r_0^2 A^{2/3}$, one expects

$$|Q| \lesssim e\,R^2 \sim 0.06\text{–}0.5\,e\cdot\mathrm{b} \tag{1.24}$$

from light to heavy nuclei. Many odd-A moments fall in that range, but rare-earth and actinide nuclei reach several $e\cdot\mathrm{b}$: the entire charged core is permanently deformed, so many protons contribute coherently [6, 2]. Even–even nuclei have vanishing spectroscopic Q in the laboratory frame if $I = 0$, but large *intrinsic* quadrupole moments show up in rotational spectra and in Coulomb excitation.

1.7.4 Tensor operator S_{12} and noncentral force

The simplest scalar operator that couples spin and relative coordinate is

$$S_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2. \tag{1.25}$$

A potential $V_T(r)\, S_{12}$ is the tensor force. It vanishes when averaged over a spherically symmetric s-wave alone, but it mixes ${}^3S_1$ and ${}^3D_1$ and generates Q_d. In heavier nuclei the same tensor force contributes to the spin-orbit splitting and to few-nucleon binding, though its expectation value is partially averaged by many-body motion [14, 6].

One-pion exchange naturally produces a tensor potential of Yukawa range $r_0 \approx 1.4\,\mathrm{fm}$; that is the long-range reason the deuteron is not a pure s-wave state [9, 14].

1.7.5 Parity and multipole selection

A static electric multipole of order ℓ has parity $(-1)^\ell$; a magnetic multipole has parity $(-1)^{\ell+1}$. Nuclear ground states of definite parity therefore forbid static electric dipoles: the lowest allowed electric deformation is the quadrupole. That is why Q is the first multipole beyond the monopole in the structural discussion of the deuteron and of deformed heavy nuclei [1, 6].

1.8 Problems with solutions

Problem 9.1 — $Q = 0$ for a sphere

Show that $Q_{zz} = \int \rho(3z^2 - r^2)\, \mathrm{d}^3r$ vanishes for any spherically symmetric $\rho(r)$.

Solution 9.1

Spherical symmetry $\Rightarrow \int x^2\rho = \int y^2\rho = \int z^2\rho = \frac{1}{3}\int r^2\rho$. Hence

$$\int (3z^2 - r^2)\rho = 3 \cdot \tfrac{1}{3} \int r^2\rho - \int r^2\rho = 0.$$

A pure $L = 0$ deuteron would have $Q_d = 0$ [6].

Problem 9.2 — Units of Q_d

The deuteron quadrupole moment is $Q_d = 0.00288\,\mathrm{b}$ and also $2.88 \times 10^{-27}\,\mathrm{cm}^2$. Convert the barn value to $\mathrm{e}\cdot\mathrm{fm}^2$ ($1\,\mathrm{b} = 100\,\mathrm{fm}^2$) and verify consistency with $10^{-27}\,\mathrm{cm}^2$.

Solution 9.2

$$0.00288\,\mathrm{b} = 0.00288 \times 100 = 0.288\,\mathrm{fm}^2 = 0.288\,\mathrm{e}\cdot\mathrm{fm}^2$$

(charge e understood in nuclear tables). Also $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2 = 10^{-24}\,\mathrm{cm}^2$, so

$$0.00288\,\mathrm{b} = 0.00288 \times 10^{-24}\,\mathrm{cm}^2 = 2.88 \times 10^{-27}\,\mathrm{cm}^2,$$

matching the lecture value. Compared with eR^2 for $R \sim 2\,\mathrm{fm}$ ($\sim 4\,e\cdot\mathrm{fm}^2$), the deformation is mild [6].

Problem 9.3 — Far-field expansion coefficient

Starting from

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\left(1 - 2\frac{r'}{r}\cos\theta\right)^{-1/2}$$

(for $r \gg r'$), expand to order $(r'/r)^2$ and identify the $P_2(\cos\theta)$ term that defines the quadrupole contribution to Φ.

Solution 9.3

Binomial expansion $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$ with $x = 2(r'/r)\cos\theta$:

$$\begin{aligned}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} &= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + \frac{3}{2}\left(\frac{r'}{r}\right)^2\cos^2\theta + \cdots\right] \\ &= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + \frac{r'^2}{2r^2}(3\cos^2\theta - 1) + \frac{r'^2}{2r^2} + \cdots\right].\end{aligned}$$

The angular piece $(3\cos^2\theta - 1) = 2P_2(\cos\theta)$ multiplies r'^2/r^3 in Φ and defines the quadrupole multipole after integration against $\rho(\boldsymbol{r}')$ [6, 1].

Problem 9.4 — Why a tensor force is required

Given $I^\pi = 1^+$, $\mu_d \neq \mu_p + \mu_n$, and $Q_d \neq 0$, explain why a purely central NN potential is ruled out.

Solution 9.4

A central potential conserves L, so each energy eigenstate has definite L. Even parity and $I = 1$ allow $L = 0$ and $L = 2$ separately, but not a coherent mixture in one eigenstate. Nonzero Q_d requires nonsphericity (D-wave), and the μ_d deficit requires the same mixture. The tensor force $V_T(r)S_{12}$ is the non-central operator that mixes ${}^3S_1$ and ${}^3D_1$ [6, 14].

Problem 9.5 — Tensor operator on a pure S-wave

The tensor operator is $S_{12} = 3(\boldsymbol{\sigma}_1\cdot\hat{r})(\boldsymbol{\sigma}_2\cdot\hat{r}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$. Argue why $\langle S_{12}\rangle = 0$ in a pure ${}^3S_1$ state with spherical average, yet S_{12} can still mix S and D waves.

Solution 9.5

In a pure s-wave the orbital density is isotropic. Averaging $(\boldsymbol{\sigma}_1\cdot\hat{r})(\boldsymbol{\sigma}_2\cdot\hat{r})$ over angles reduces it to $\frac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, so $\langle S_{12}\rangle = 0$. Off-diagonal matrix elements between $L = 0$ and $L = 2$ need not vanish: S_{12} transforms as a rank-2 tensor and connects $\Delta L = 2$. That is how a small D-wave is generated even though the pure-S diagonal expectation is zero [14, 6].

Problem 9.6 — Geometric scale of Q

Estimate the single-particle quadrupole scale eR^2 for $R = r_0 A^{1/3}$ with $r_0 = 1.2\,\text{fm}$ at $A = 2$ and at $A = 160$. Compare with $Q_d = 0.288\,e \cdot \text{fm}^2$ and with rare-earth moments of several $e \cdot \text{b}$.

Solution 9.6

$$R(A{=}2) \approx 1.2 \times 1.26 \approx 1.5\,\text{fm}, \quad eR^2 \approx 2.3\,e \cdot \text{fm}^2;$$

$$R(A{=}160) \approx 1.2 \times 5.4 \approx 6.5\,\text{fm}, \quad eR^2 \approx 42\,e \cdot \text{fm}^2 = 0.42\,e \cdot \text{b}.$$

Q_d is $\ll eR^2(A{=}2)$ (weak D-wave deformation). Collective rare-earth $Q \sim$ few b exceed the single-particle scale by an order of magnitude (coherent core deformation) [6, 2].

Bibliography

[1] R. A. Dunlap, *An Introduction to the Physics of Nuclei and Particles* (2nd ed.), IOP Publishing, Bristol (2023), DOI: 10.1088/978-0-7503-6094-4, Ch. 3 (Nuclear composition and size).

[2] W. Demtröder, *Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2023).

[3] S. Petrera, *Problems and Solutions in Nuclear and Particle Physics*, UNITEXT for Physics, Springer, Cham (2019).

[4] F. Terranova, *A Modern Primer in Particle and Nuclear Physics*, Oxford University Press (2021).

[5] S. D'Auria, *Introduction to Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2018).

[6] K. S. Krane, *Introductory Nuclear Physics*, Wiley, New York (1988).

[7] R. Hofstadter, "Electron scattering and nuclear structure," Rev. Mod. Phys. **28**, 214 (1956).

[8] H. F. Ehrenberg, R. Hofstadter, U. Meyer-Berkhout, D. G. Ravenhall, and S. E. Sobottka, "High-Energy Electron Scattering and the Charge Distribution of Carbon-12 and Oxygen-16," Phys. Rev. **113**, 666 (1959).

[9] H. Yukawa, "On the interaction of elementary particles. I," Proc. Phys.-Math. Soc. Japan **17**, 48 (1935).

[10] R. D. Woods and D. S. Saxon, "Diffuse surface optical model for nucleon-nuclei scattering," Phys. Rev. **95**, 577 (1954).

[11] H. G. J. Moseley, "The high-frequency spectra of the elements," Phil. Mag. **26**, 1024 (1913); **27**, 703 (1914).

[12] C. F. von Weizsäcker, "Zur Theorie der Kernmassen," Z. Phys. **96**, 431 (1935).

[13] P. Cappellaro, *Introduction to Applied Nuclear Physics* (MIT OpenCourseWare / LibreTexts), CC BY-NC-SA 4.0.

[14] J.-L. Basdevant, J. Rich, and M. Spiro, *Fundamentals in Nuclear Physics: From Nuclear Structure to Cosmology*, Springer, New York (2005).

[15] D. Tong, *Particle Physics*, Lecture notes (University of Cambridge / CERN school notes).