Chapter 1

Deuteron Spin and Magnetic Moment

1.1 Nuclear spin and parity: careful definitions

The binding energy of the deuteron fixes the radial scale of a central NN well, under the assumption that the relative orbital angular momentum is $\ell = 0$. That assumption must now be tested with every other measured property of the ground state: the total angular momentum (nuclear spin), the parity, and the magnetic dipole moment [6, 2].

1.1.1 What "nuclear spin" means

Each nucleon carries orbital angular momentum $\boldsymbol{\ell}$ and intrinsic spin $\boldsymbol{s} = \frac{1}{2}\hbar$ (in units where we often set $\hbar = 1$ for quantum numbers). Their sum is

$$\boldsymbol{j} = \boldsymbol{\ell} + \boldsymbol{s}. \tag{1.1}$$

In a nucleus the total angular momentum is the vector sum of all nucleon contributions. Nuclear physicists call this total $\boldsymbol{I}$ and name it the **nuclear spin**, even though it includes orbital motion as well as spin. The usual quantum rules apply:

$$\boldsymbol{I}^2 \to I(I+1)\hbar^2, \qquad I_z \to M_I\hbar, \quad M_I = -I, \ldots, +I. \tag{1.2}$$

When two angular momenta j_1 and j_2 are coupled, the allowed total values follow the triangle rule

$$J = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2. \tag{1.3}$$

This rule will be applied twice: first $\boldsymbol{S} = \boldsymbol{s}_p + \boldsymbol{s}_n$, then $\boldsymbol{I} = \boldsymbol{L} + \boldsymbol{S}$.

1.1.2 Parity

Parity is the behaviour of the spatial wave function under $\boldsymbol{r} \to -\boldsymbol{r}$. If $\psi(-\boldsymbol{r}) = +\psi(\boldsymbol{r})$ the parity is even ($\pi = +$); if $\psi(-\boldsymbol{r}) = -\psi(\boldsymbol{r})$ it is odd ($\pi = -$). A central-potential wave function separates as

$$\psi_{\ell m}(\boldsymbol{r}) = R_\ell(r) Y_{\ell m}(\theta, \phi). \tag{1.4}$$

Inversion leaves $r = |\boldsymbol{r}|$ unchanged but sends $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$. Spherical harmonics obey

$$Y_{\ell m}(\pi - \theta, \phi + \pi) = (-1)^\ell Y_{\ell m}(\theta, \phi). \tag{1.5}$$

Therefore

$$\psi_{\ell m}(-\boldsymbol{r}) = (-1)^\ell \psi_{\ell m}(\boldsymbol{r}), \qquad \pi = (-1)^\ell. \tag{1.6}$$

Even ℓ gives even parity; odd ℓ gives odd parity.

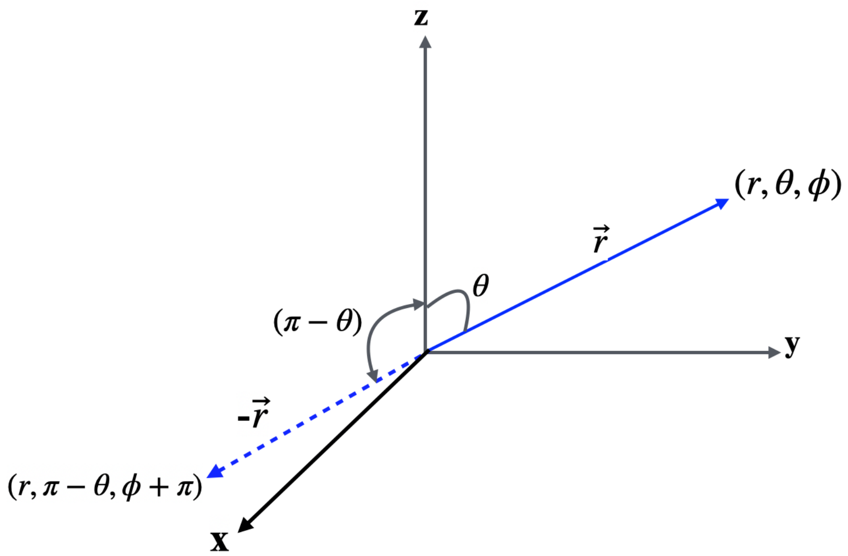

Figure 1.1: Inversion $\boldsymbol{r} \leftrightarrow -\boldsymbol{r}$. Nuclear states have definite parity when the Hamiltonian conserves parity.

1.1.3 Experimental assignment for the deuteron

Experiment gives

$$I^\pi(^2\mathrm{H}) = 1^+. \tag{1.7}$$

That single label will eliminate most of the angular-momentum possibilities we might have imagined.

1.2 Coupling two spins: four ways to make $I = 1$

The deuteron total angular momentum is

$$\boldsymbol{I} = \boldsymbol{s}_p + \boldsymbol{s}_n + \boldsymbol{L}, \tag{1.8}$$

where $\boldsymbol{L}$ is the relative orbital angular momentum. Each nucleon has $s = \frac{1}{2}$, so the total spin $\boldsymbol{S} = \boldsymbol{s}_p + \boldsymbol{s}_n$ obeys

$$S = \left|\tfrac{1}{2} - \tfrac{1}{2}\right|, \ldots, \tfrac{1}{2} + \tfrac{1}{2} = 0,\ 1. \tag{1.9}$$

For fixed S and L, the second application of Equation (1.3) gives

$$|L - S| \leq I \leq L + S. \tag{1.10}$$

We require $I = 1$:

- If $S = 0$, then $I = L$, so only $L = 1$ works.
- If $S = 1$, then $|L - 1| \leq 1 \leq L + 1$, which permits $L = 0, 1, 2$.

Thus there are exactly four couplings [6]:

(a) spins parallel ($S = 1$) with $L = 0$;

(b) spins antiparallel ($S = 0$) with $L = 1$;

(c) spins parallel ($S = 1$) with $L = 1$;

(d) spins parallel ($S = 1$) with $L = 2$.

Parity decides among them. The deuteron has *even* parity, and orbital parity is $(-1)^L$. Therefore L must be even: $L = 1$ is ruled out. Cases (b) and (c) drop. We are left with

$$^3S_1 \quad (L = 0,\, S = 1) \qquad \text{and} \qquad ^3D_1 \quad (L = 2,\, S = 1). \tag{1.11}$$

The bound state is **100% spin triplet**. There is no bound singlet pn state.

The notation $^{2S+1}L_I$ now follows directly. The superscript is the spin multiplicity $2S+1$, the letter denotes $L = 0, 1, 2, \ldots \to S, P, D, \ldots$, and the subscript is the total angular momentum I. Hence the two even-parity possibilities are $^3S_1$ and $^3D_1$.

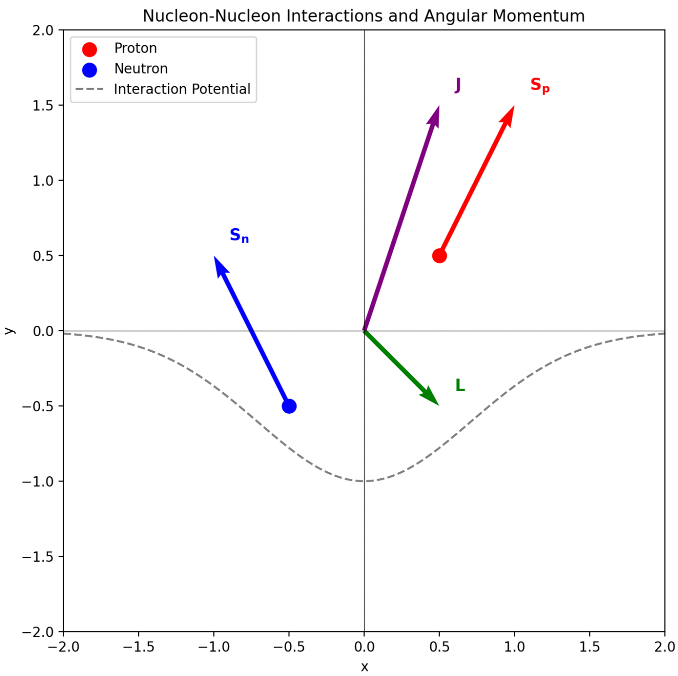

Figure 1.2: Schematic spins and orbital angular momentum in the NN system. The deuteron ground state is dominated by parallel spins and $L = 0$, with a small $L = 2$ admixture.

Key idea

Spin dependence of the nuclear force. Nature binds the triplet channel and not the singlet. The NN force depends on the relative orientation of the nucleon spins. A spin-independent central well would not explain that fact.

1.2.1 Triplet and singlet spin states

Start with the unique highest-projection product state,

$$|1,1\rangle = |\uparrow\uparrow\rangle. \tag{1.12}$$

Apply the total lowering operator $S_- = s_{p-} + s_{n-}$. Since $S_-|1,1\rangle = \hbar\sqrt{2}\,|1,0\rangle$,

$$S_-|\uparrow\uparrow\rangle = \hbar|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle, \tag{1.13}$$

$$\therefore \quad |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right). \tag{1.14}$$

A second lowering gives $|1,-1\rangle = |\downarrow\downarrow\rangle$. The three triplet states ($S = 1$) are therefore

$$|1,1\rangle = |\uparrow\uparrow\rangle, \qquad |1,0\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \qquad |1,-1\rangle = |\downarrow\downarrow\rangle. \tag{1.15}$$

The remaining normalized $M_S = 0$ combination must be orthogonal to $|1,0\rangle$; it is the singlet

$$|0,0\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{1.16}$$

A useful operator that distinguishes them is $\boldsymbol{s}_p \cdot \boldsymbol{s}_n$. Square $\boldsymbol{S} = \boldsymbol{s}_p + \boldsymbol{s}_n$:

$$\boldsymbol{S}^2 = \boldsymbol{s}_p^2 + \boldsymbol{s}_n^2 + 2\boldsymbol{s}_p \cdot \boldsymbol{s}_n, \tag{1.17}$$

$$\therefore \quad \boldsymbol{s}_p \cdot \boldsymbol{s}_n = \frac{1}{2}\left(\boldsymbol{S}^2 - \boldsymbol{s}_p^2 - \boldsymbol{s}_n^2\right). \tag{1.18}$$

Because $s_p = s_n = \frac{1}{2}$, $\boldsymbol{s}_p^2 = \boldsymbol{s}_n^2 = \frac{3}{4}\hbar^2$. Thus

$$\langle \boldsymbol{s}_p \cdot \boldsymbol{s}_n \rangle = \frac{\hbar^2}{2}\left[S(S+1) - \frac{3}{4} - \frac{3}{4}\right] = \begin{cases} +\dfrac{\hbar^2}{4}, & S = 1, \\ -\dfrac{3\hbar^2}{4}, & S = 0. \end{cases} \tag{1.19}$$

A potential containing $\boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_n$ can therefore deepen the triplet well relative to the singlet. That is why the square-well depth of Lecture 7 applies to the *triplet* channel; the singlet well is shallower and does not bind.

Caution

The singlet value is not zero. It is negative ($-3\hbar^2/4$). Saying the singlet product "vanishes" is a common slip; the correct eigenvalues matter when one writes $V_\sigma(r)$.

1.3 Magnetic moments: from classical orbits to g-factors

1.3.1 Nuclear magneton

Consider a charge e moving with speed v in a circular orbit of radius r. Its period, current, and loop area are

$$T = \frac{2\pi r}{v}, \qquad \mathcal{I} = \frac{e}{T} = \frac{ev}{2\pi r}, \qquad \mathcal{A} = \pi r^2. \tag{1.20}$$

The magnetic dipole moment is current times area,

$$\mu = \mathcal{I}\mathcal{A} = \frac{evr}{2}. \tag{1.21}$$

Since the orbital angular momentum is $L = mvr$, this becomes

$$\boldsymbol{\mu}_\ell = \frac{e}{2m}\boldsymbol{L}. \tag{1.22}$$

For a proton, set $m = m_p$ and factor out one quantum $\hbar$. This defines the **nuclear magneton**

$$\mu_N = \frac{e\hbar}{2m_p} \approx 3.1525 \times 10^{-8}\,\mathrm{eV/T}. \tag{1.23}$$

It is smaller than the Bohr magneton by $m_e/m_p \approx 1/1836$. Ordinary magnetism of matter is electronic; nuclear magnetism is a delicate probe [6].

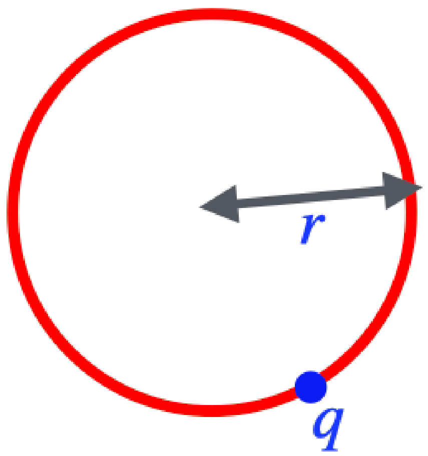

Figure 1.3: Classical orbital current: charge on a loop produces $\mu \propto qL/(2m)$. Neutrons have $q = 0$ and no orbital moment from net charge.

1.3.2 Orbital and spin contributions

We write

$$\mu_\ell = g_\ell\,\mu_N\,\frac{\ell_z}{\hbar}, \qquad \mu_s = g_s\,\mu_N\,\frac{s_z}{\hbar}. \tag{1.24}$$

For proton orbital motion $g_\ell = 1$; for the neutron $g_\ell = 0$. For free nucleon *spins*, experiment finds anomalous g-factors far from the Dirac value $g_s = 2$:

$$g_p = 5.5856912, \tag{1.25}$$

$$g_n = -3.8260837. \tag{1.26}$$

Equivalently, $\mu_p = +2.7928\,\mu_N$ and $\mu_n = -1.9130\,\mu_N$. The neutron's nonzero moment and the proton's large g_p are early evidence that nucleons are composite (meson clouds; today, quarks) [6, 2].

1.4 Deuteron magnetic moment: pure S-wave calculation

If the orbital angular momentum vanishes, there is no orbital magnetic moment: only the intrinsic spins of the proton and neutron contribute. With $I^\pi = 1^+$ and a pure ${}^3S_1$ configuration, the stretched state $M_I = 1$ has both spins aligned ($s_z = +\hbar/2$). The deuteron moment is then simply the sum of the free-nucleon moments [6],

$$\mu_d(L{=}0) = \mu_p + \mu_n = \frac{g_p + g_n}{2}\,\mu_N = \frac{5.5857 - 3.8261}{2}\,\mu_N = 0.8798\,\mu_N. \tag{1.27}$$

Experiment finds

$$\mu_d^{(\mathrm{exp})} = 0.8574376(4)\,\mu_N, \tag{1.28}$$

about $0.022\,\mu_N$ ($\sim 2.5\%$) below the pure S-wave value. The agreement is already striking: most of μ_d is indeed $\mu_p + \mu_n$. The residual deficit, however, is a genuine structural effect. Meson-exchange currents and relativistic corrections can shift the moment, but within the nonrelativistic NN picture the natural explanation is a small admixture of $L = 2$ in the ground-state wave function [6].

Key idea

Physics of the discrepancy. A pure s-wave deuteron would have $\mu_d = \mu_p + \mu_n$. The measured moment is slightly smaller, signalling a few-percent D-wave component and, with it, a non-central piece of the force.

1.5 D-wave admixture from μ_d

Write the ground state as a coherent mixture of even orbital waves allowed by $I^\pi = 1^+$,

$$|\psi\rangle = a_S\,|L{=}0\rangle + a_D\,|L{=}2\rangle, \qquad |a_S|^2 + |a_D|^2 = 1. \tag{1.29}$$

1.5.1 Expectation value of the magnetic-moment operator

The numerical D-state moment should not simply be quoted; it follows from an angular-momentum expectation value. In the centre-of-mass frame, and neglecting exchange currents, the z-component of the two-nucleon magnetic-moment operator is

$$\frac{\hat{\mu}_z}{\mu_N} = \frac{1}{2}\frac{\hat{L}_z}{\hbar} + g_p \frac{\hat{s}_{pz}}{\hbar} + g_n \frac{\hat{s}_{nz}}{\hbar}. \tag{1.30}$$

Only the proton contributes an orbital magnetic moment. For nearly equal nucleon masses its orbital angular momentum about the centre of mass can be obtained explicitly. Let $M = m_p + m_n$, $\boldsymbol{r} = \boldsymbol{r}_p - \boldsymbol{r}_n$, and let $\boldsymbol{p}$ be the relative momentum in the centre-of-mass frame. Then

$$\boldsymbol{r}_p = \frac{m_n}{M}\boldsymbol{r}, \qquad \boldsymbol{p}_p = \boldsymbol{p}, \tag{1.31}$$

so

$$\boldsymbol{L}_p = \boldsymbol{r}_p \times \boldsymbol{p}_p = \frac{m_n}{M}\,\boldsymbol{r} \times \boldsymbol{p} = \frac{m_n}{M}\boldsymbol{L} \simeq \frac{1}{2}\boldsymbol{L}. \tag{1.32}$$

This produces the first term of Equation (1.30). Within the spin-triplet state the proton and neutron share the total spin symmetrically. Equivalently, the antisymmetric operator $\boldsymbol{s}_p - \boldsymbol{s}_n$ has zero diagonal matrix element between symmetric triplet spin states, and therefore

$$\langle s_{pz} \rangle = \langle s_{nz} \rangle = \frac{1}{2}\langle S_z \rangle. \tag{1.33}$$

The projection theorem itself follows in two steps. First, $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ implies

$$\boldsymbol{L} \cdot \boldsymbol{J} = \boldsymbol{L} \cdot (\boldsymbol{L} + \boldsymbol{S}) = \boldsymbol{L}^2 + \boldsymbol{L} \cdot \boldsymbol{S}, \tag{1.34}$$

$$\boldsymbol{J}^2 = \boldsymbol{L}^2 + \boldsymbol{S}^2 + 2\boldsymbol{L} \cdot \boldsymbol{S}. \tag{1.35}$$

Eliminating $\boldsymbol{L} \cdot \boldsymbol{S}$ gives

$$\boldsymbol{L} \cdot \boldsymbol{J} = \frac{1}{2}\left(\boldsymbol{J}^2 + \boldsymbol{L}^2 - \boldsymbol{S}^2\right). \tag{1.36}$$

In a state $|LSJM\rangle$, rotational symmetry says that the expectation value of $\boldsymbol{L}$ can point only along $\boldsymbol{J}$. Its z-projection is consequently

$$\langle L_z \rangle = \frac{\langle \boldsymbol{L} \cdot \boldsymbol{J} \rangle}{J(J+1)\hbar^2}\,\langle J_z \rangle. \tag{1.37}$$

Using $\langle J_z \rangle = M\hbar$ and the eigenvalues of the squared angular momenta in Equation (1.36) gives

$$\langle L_z \rangle = M\hbar\,\frac{J(J+1) + L(L+1) - S(S+1)}{2J(J+1)}, \tag{1.38}$$

$$\langle S_z \rangle = M\hbar\,\frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}. \tag{1.39}$$

The second line can be obtained either by interchanging L and S or from $\langle S_z \rangle = M\hbar - \langle L_z \rangle$. The observable deuteron moment is evaluated in the stretched state $J = M = 1$.

For $^3S_1$, $L = 0$ and $S = 1$, so

$$\frac{\langle L_z \rangle}{\hbar} = 1\,\frac{1(1+1) + 0(0+1) - 1(1+1)}{2(1)(1+1)} = \frac{2+0-2}{4} = 0, \tag{1.40}$$

$$\frac{\langle S_z \rangle}{\hbar} = 1\,\frac{1(1+1) + 1(1+1) - 0(0+1)}{2(1)(1+1)} = \frac{2+2-0}{4} = 1. \tag{1.41}$$

Equation (1.30) therefore reproduces

$$\begin{aligned} \mu(^3S_1) &= \left[\frac{1}{2}(0) + \frac{g_p + g_n}{2}(1)\right]\mu_N \\ &= \frac{g_p + g_n}{2}\,\mu_N = \mu_p + \mu_n = 0.8798\,\mu_N. \end{aligned} \tag{1.42}$$

For $^3D_1$, $L = 2$, $S = 1$, and $J = M = 1$. Equations (1.38)–(1.39) give, step by step,

$$\frac{\langle L_z \rangle}{\hbar} = 1\,\frac{1(1+1) + 2(2+1) - 1(1+1)}{2(1)(1+1)} = \frac{2+6-2}{4} = \frac{3}{2}, \tag{1.43}$$

$$\frac{\langle S_z \rangle}{\hbar} = 1\,\frac{1(1+1) + 1(1+1) - 2(2+1)}{2(1)(1+1)} = \frac{2+2-6}{4} = -\frac{1}{2}. \tag{1.44}$$

The orbital and spin angular momenta are therefore partly antiparallel in the D state. Substitution into (1.30) yields

$$\begin{aligned} \mu(^3D_1) &= \left[\frac{1}{2}\left(\frac{3}{2}\right) + \frac{g_p + g_n}{2}\left(-\frac{1}{2}\right)\right]\mu_N \\ &= \frac{1}{4}(3 - g_p - g_n)\,\mu_N \approx 0.310\,\mu_N. \end{aligned} \tag{1.45}$$

This is the expectation-value derivation of the smaller pure-D magnetic moment [6, 2].

In the one-body operator approximation, $\langle ^3S_1|\hat{\mu}_z|^3D_1\rangle = 0$: the angular functions with $L = 0$ and $L = 2$ are orthogonal, and the operator in (1.30) does not change L. The mixed-state expectation value can be expanded before the cross terms are discarded:

$$\begin{aligned} \langle\psi|\hat{\mu}_z|\psi\rangle &= |a_S|^2\langle ^3S_1|\hat{\mu}_z|^3S_1\rangle + |a_D|^2\langle ^3D_1|\hat{\mu}_z|^3D_1\rangle \\ &\quad + a_S^* a_D\langle ^3S_1|\hat{\mu}_z|^3D_1\rangle + a_D^* a_S\langle ^3D_1|\hat{\mu}_z|^3S_1\rangle \\ &= P_S\mu_S + P_D\mu_D. \end{aligned} \tag{1.46}$$

Therefore

$$\mu_d = P_S\,(0.880\,\mu_N) + P_D\,(0.310\,\mu_N), \qquad P_S + P_D = 1. \tag{1.47}$$

To obtain the commonly used form, substitute $P_S = 1 - P_D$ and $\mu_D = \frac{1}{4}(3 - g_p - g_n)\mu_N$:

$$\begin{aligned} \mu_d &= (1 - P_D)(\mu_p + \mu_n) + P_D\frac{3 - g_p - g_n}{4}\mu_N \\ &= (\mu_p + \mu_n) + P_D\left[\frac{3}{4}\mu_N - \frac{3}{2}(\mu_p + \mu_n)\right] \\ &= (\mu_p + \mu_n) - \frac{3}{2}\left[\mu_p + \mu_n - \frac{1}{2}\mu_N\right] P_D. \end{aligned} \tag{1.48}$$

Finally, requiring $\mu_d = 0.8574\,\mu_N$ and retaining the displayed rounded moments gives

$$\begin{aligned} 0.8574 &= 0.8798(1-P_D) + 0.310P_D \\ &= 0.8798 - (0.8798-0.310)P_D, \end{aligned} \tag{1.49}$$

$$\begin{aligned} (0.8798-0.310)P_D &= 0.8798 - 0.8574, \\ P_D &= \frac{0.8798-0.8574}{0.8798-0.310} = 0.0393 \approx 0.039, \qquad P_S = 1 - P_D \approx 0.961. \end{aligned} \tag{1.50}$$

This classroom estimate suggests a few-percent D-wave. It is *not* a model-independent measurement of P_D: meson-exchange currents and relativistic corrections also shift μ_d. Independent NN scattering analyses still find a few-percent D-state probability, so the qualitative conclusion is robust: the force cannot be purely central. Only a tensor interaction can mix $L = 0$ with $L = 2$ in a single eigenstate. The electric quadrupole moment $Q_d \neq 0$ (Lecture 9) establishes that mixing far more cleanly than the magnetic-moment deficit alone [6].

Caution

Do not promote $P_D \approx 4\%$ to an observable. Treat it as a pedagogical estimate that motivates looking for a tensor force; the decisive geometric evidence is the nonzero quadrupole moment.

Key idea

What Lectures 7–9 establish together. Weak binding fixes the depth and range of the residual force. Spin and parity fix the channel as mostly ${}^3S_1$. The magnetic-moment deficit *suggests* a small D-wave; the quadrupole moment *requires* one, and with it a tensor force.

1.6 Preview of heavier nuclei

In odd-A nuclei, pairing often cancels the moments of all but the last unpaired nucleon. Plotting μ versus I produces the Schmidt lines. Real nuclei fall between those lines because of configuration mixing, the many-body cousin of the deuteron's D-wave. The deuteron is the cleanest few-body example of the same idea: free moments first, then a small admixture [6, 1].

Takeaway

Takeaways (Lecture 8).

- $I^\pi = 1^+$: even parity forces even L; the bound state is spin triplet.
- Spin-dependent force: triplet binds, singlet does not.
- Pure S-wave $\mu_d = 0.880\,\mu_N$; experiment $0.857\,\mu_N$.
- Mixture $\sim 96\%\ S + \sim 4\%\ D$ is a classroom estimate of the magnetic deficit; quadrupole data (next lecture) establish the D-wave and the tensor force.

1.7 Deeper physics: magnetic moments beyond the deuteron

1.7.1 How μ is measured (Rabi and NMR)

Place a state of angular momentum I in a uniform field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. With $\hat{\mu}_z = g_I \mu_N \hat{I}_z/\hbar$, the Zeeman Hamiltonian is

$$\hat{H}_Z = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B} = -g_I \mu_N B \frac{\hat{I}_z}{\hbar}. \tag{1.51}$$

Because $\hat{I}_z|I, M_I\rangle = M_I \hbar |I, M_I\rangle$,

$$E_{M_I} = -g_I \mu_N B M_I. \tag{1.52}$$

Adjacent substates therefore differ by

$$|\Delta E| = g_I \mu_N B = h\nu. \tag{1.53}$$

The Rabi molecular-beam method and nuclear magnetic resonance detect the frequency ν at which a radiofrequency field drives $\Delta M_I = \pm 1$ transitions. Since B is known, the resonance gives g_I, and the stretched-state moment is $\mu = g_I I \mu_N$. Free-nucleon moments $\mu_p = +2.7928\,\mu_N$ and $\mu_n = -1.9130\,\mu_N$ are among the most accurately known quantities in nuclear physics [2, 1].

1.7.2 Schmidt lines (preview of the shell model)

For odd-A nuclei the magnetic moment is often dominated by the last unpaired nucleon. Plotting μ versus I produces the Schmidt lines: two limiting curves for $j = \ell \pm \frac{1}{2}$. Real nuclei fall between the lines because of configuration mixing and meson-exchange currents. The deuteron is the few-body cousin of that story: free $\mu_p + \mu_n$ is the extreme independent-particle limit; a few-percent D-wave admixture is the simplest configuration mixing [6, 1]. The two Schmidt formulae are derived, rather than merely quoted, in Section 1.8.2.

1.7.3 Isospin language

Proton and neutron form an isospin doublet $t = \frac{1}{2}$. The deuteron has total isospin $T = 0$, and this assignment follows from exchange symmetry. For two nucleons, the spatial, spin, and isospin factors acquire under exchange the signs

$$(-1)^L, \qquad (-1)^{S+1}, \qquad (-1)^{T+1}, \tag{1.54}$$

respectively. Their product must be -1 for fermions, so

$$(-1)^{L+S+T} = -1. \tag{1.55}$$

The deuteron has even L and $S = 1$. Hence $L + S + T$ is odd only when $T = 0$. Explicitly, its isospin wave function is the antisymmetric combination

$$|T = 0, T_z = 0\rangle = \frac{1}{\sqrt{2}} \left(|pn\rangle - |np\rangle\right). \tag{1.56}$$

The $T = 1$, $S = 0$ pn partner of the virtual ${}^1S_0$ state is unbound, as are free nn and pp (after Coulomb). Charge independence of the strong force organises NN scattering lengths into a nearly universal pattern once Coulomb is removed. Lecture 10 develops this NN spin–isospin structure through phase shifts [4].

Remark

Quarks and anomalous moments. Dirac particles with charge e and mass m have $g = 2$. The proton's $g_p \approx 5.59$ and the neutron's nonzero moment are direct evidence of composite structure (charged quark currents and meson clouds), already anticipated in Lecture 7's residual-force discussion [2, 4].

1.7.4 Beyond the one-body magnetic operator

For the one-body operator derived in Section 1.5.1, the S–D cross term vanishes: $\hat{L}_z$, $\hat{s}_{pz}$, and $\hat{s}_{nz}$ do not change L, and the $L = 0$ and $L = 2$ angular functions are orthogonal. This gives the probability-weighted expectation value in Equation (1.48).

A precision calculation must add two-body meson-exchange currents, relativistic corrections, and small nucleon-structure effects. Those contributions alter the magnetic moment without being equivalent to a change in P_D. Modern potentials still find a few-percent D-state probability, so the classroom estimate is useful, but P_D itself is model dependent [6].

1.8 Further physics: magnetic moments and the nuclear magneton

To each angular momentum is associated a magnetic dipole

$$\boldsymbol{\mu} = g\,\mu_N \frac{\boldsymbol{j}}{\hbar}, \qquad \mu_N = \frac{e\hbar}{2m_p}. \tag{1.57}$$

For orbital motion of a proton, $g_\ell = 1$; for a neutron, $g_\ell = 0$. Spin g-factors of free nucleons are anomalous ($g_p \approx 5.59$, $g_n \approx -3.83$), a signal of internal structure [13, 6, 2, 5].

In odd-A nuclei a first approximation attributes μ to the last unpaired nucleon (Schmidt estimate). The deuteron is the clean two-body limit of the same idea: free moments first, then a small configuration admixture (D-wave) to match experiment. Exact g-factors of complex nuclei require interactions among many nucleons; the deuteron isolates the few-body physics.

1.8.1 Spin dependence from the deuteron alone

From $I^\pi = 1^+$ and $\mu_d \approx \mu_p + \mu_n$ one concludes that the ground state is mostly ${}^3S_1$. The absence of a bound ${}^1S_0$ pn state shows that the force is spin-dependent: both channels are attractive, but only the triplet is deep enough to bind [14, 6]. The small quadrupole moment and the μ_d deficit then require a few-percent ${}^3D_1$ admixture (tensor force; Lecture 9).

1.8.2 Schmidt lines for odd-A nuclei

For a single valence nucleon with orbital angular momentum ℓ and spin $s = \frac{1}{2}$, the one-body magnetic operator is

$$\frac{\hat{\mu}_z}{\mu_N} = g_\ell \frac{\hat{\ell}_z}{\hbar} + g_s \frac{\hat{s}_z}{\hbar}. \tag{1.58}$$

The measured moment is the expectation value in the stretched state $|j, m = j\rangle$. Since $j_z = \ell_z + s_z$, it is useful to write

$$\frac{\mu}{\mu_N} = g_\ell j + (g_s - g_\ell)\frac{\langle s_z \rangle}{\hbar}. \tag{1.59}$$

The projection theorem derived in Section 1.5.1, with $L \to \ell$, $S \to s$, and $M = j$, gives

$$\frac{\langle s_z \rangle}{\hbar} = j\,\frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}. \tag{1.60}$$

For aligned spin, $j = \ell + \frac{1}{2}$, so $\ell = j - \frac{1}{2}$ and $s(s+1) = \frac{3}{4}$. Substitution into Equation (1.60) gives

$$\begin{aligned}\frac{\langle s_z \rangle}{\hbar} &= \frac{j(j+1) + \frac{3}{4} - (j - \frac{1}{2})(j + \frac{1}{2})}{2(j+1)} \\ &= \frac{j^2 + j + \frac{3}{4} - (j^2 - \frac{1}{4})}{2(j+1)} = \frac{j+1}{2(j+1)} = \frac{1}{2}.\end{aligned} \tag{1.61}$$

Equation (1.59) then yields the upper Schmidt line,

$$\boxed{\frac{\mu}{\mu_N} = g_\ell j + \frac{1}{2}(g_s - g_\ell), \qquad j = \ell + \frac{1}{2}.} \tag{1.62}$$

Equivalently, $\mu/\mu_N = g_\ell \ell + g_s/2$: in the stretched state the orbital and spin projections are simply ℓ and $1/2$.

For anti-aligned spin, $j = \ell - \frac{1}{2}$, so $\ell = j + \frac{1}{2}$. Now

$$\begin{aligned}\frac{\langle s_z \rangle}{\hbar} &= \frac{j(j+1) + \frac{3}{4} - (j + \frac{1}{2})(j + \frac{3}{2})}{2(j+1)} \\ &= \frac{j^2 + j + \frac{3}{4} - (j^2 + 2j + \frac{3}{4})}{2(j+1)} = -\frac{j}{2(j+1)}.\end{aligned} \tag{1.63}$$

The lower Schmidt line is therefore

$$\boxed{\frac{\mu}{\mu_N} = g_\ell j - (g_s - g_\ell)\frac{j}{2(j+1)}, \qquad j = \ell - \frac{1}{2}.} \tag{1.64}$$

For an unpaired proton one inserts $g_\ell = 1$ and $g_s = g_p$; for an unpaired neutron, $g_\ell = 0$ and $g_s = g_n$. These are independent particle limits, not exact predictions. Experimental moments of odd-A nuclei generally lie between the Schmidt lines, shifted toward smaller $|\mu|$ by configuration mixing and meson-exchange currents [6, 2, 13]. The deuteron is the few-body prototype of that programme: start from free g-factors, then correct for orbital admixtures.

1.8.3 Nuclear magneton vs Bohr magneton

Because $m_p \approx 1836\, m_e$,

$$\begin{aligned}\mu_N &= \frac{m_e}{m_p}\,\mu_B \\ &= \frac{m_e}{m_p}\left(\frac{e\hbar}{2m_e}\right) = \frac{e\hbar}{2m_p} \\ &\approx \frac{1}{1836}\,\mu_B. \end{aligned} \tag{1.65}$$

Nuclear moments are therefore three orders of magnitude smaller than electronic moments, which is why nuclear magnetism is a precision hyperfine effect in atoms and a separate experimental industry (NMR, atomic beams, ion traps) [5, 2].

1.9 Problems with solutions

Problem 8.1 — Pure S-wave magnetic moment

Using $g_p = 5.5857$ and $g_n = -3.8261$, compute the pure ${}^3S_1$ prediction for μ_d and the fractional difference from the experimental value $0.8574\,\mu_N$.

Solution 8.1

$$\mu_d(L{=}0) = \frac{g_p + g_n}{2}\,\mu_N = \frac{5.5857 - 3.8261}{2}\,\mu_N = 0.8798\,\mu_N.$$

$$\Delta\mu = 0.8798 - 0.8574 = 0.0224\,\mu_N, \qquad \frac{\Delta\mu}{\mu_d(L{=}0)} \approx 2.5\%.$$

Most of μ_d is $\mu_p + \mu_n$; the small deficit is structural [6].

Problem 8.2 — D-wave probability from μ_d

Assume

$$\mu_d = P_S\,(0.8798\,\mu_N) + P_D\,(0.310\,\mu_N), \qquad P_S + P_D = 1,$$

and $\mu_d = 0.8574\,\mu_N$. Find P_D.

Solution 8.2

$$0.8574 = 0.8798(1 - P_D) + 0.310\,P_D = 0.8798 - 0.5698\,P_D,$$

$$P_D = \frac{0.8798 - 0.8574}{0.5698} \approx 0.039 \approx 4\%, \qquad P_S \approx 96\%.$$

[6, 14].

Problem 8.3 — Why $L = 1$ is excluded

List (S, L) combinations that can give $I = 1$ for pn and show which survive the experimental parity $\pi = +$.

Solution 8.3

Coupling the two spin-$\frac{1}{2}$ nucleons first gives

$$S = \left|\tfrac{1}{2} - \tfrac{1}{2}\right|, \ldots, \tfrac{1}{2} + \tfrac{1}{2} = 0,\ 1.$$

Angular-momentum addition then requires

$$|L - S| \leq I \leq L + S.$$

For $S = 0$, this reduces to $I = L$, so $I = 1$ requires $L = 1$. For $S = 1$, requiring $I = 1$ gives

$$|L - 1| \leq 1 \leq L + 1,$$

which is satisfied by $L = 0, 1, 2$. Thus the four possibilities are

$$(S, L) = (0, 1),\ (1, 0),\ (1, 1),\ (1, 2).$$

Parity is $\pi = (-1)^L$. The measured $\pi = +$ eliminates both $L = 1$ cases, leaving $(S, L) = (1, 0)$ and $(1, 2)$, or ${}^3S_1$ and ${}^3D_1$. Both have $S = 1$, so the bound state is entirely spin triplet even though it is not entirely S-wave [6].

Problem 8.4 — Nuclear magneton scale

Compute $\mu_N = e\hbar/(2m_p)$ in MeV/T using $e\hbar/(2m_e) = \mu_B = 5.788 \times 10^{-11}$ MeV/T and $m_p/m_e = 1836$. Why is nuclear magnetism hard to measure compared with electronic magnetism?

Solution 8.4

$$\mu_N = \frac{m_e}{m_p}\,\mu_B = \frac{5.788 \times 10^{-11}}{1836} \approx 3.15 \times 10^{-14}\,\text{MeV/T}.$$

Nuclear moments are ~ 2000 times smaller than electronic ones, so they appear only as tiny hyperfine shifts or require specialised methods (Rabi beams, NMR) [5, 2].

Problem 8.5 — Derive the ${}^3D_1$ magnetic moment

For a pure ${}^3D_1$ state, use the projection theorem to calculate $\langle L_z \rangle$ and $\langle S_z \rangle$ in the stretched state $J = M = 1$. Then evaluate

$$\frac{\hat{\mu}_z}{\mu_N} = \frac{1}{2}\frac{\hat{L}_z}{\hbar} + \frac{g_p + g_n}{2}\frac{\hat{S}_z}{\hbar}$$

and obtain the numerical moment using the g-factors of Problem 8.1.

Solution 8.5

For $L = 2$, $S = 1$, and $J = M = 1$,

$$\begin{aligned}\frac{\langle L_z \rangle}{\hbar} &= M\frac{J(J+1) + L(L+1) - S(S+1)}{2J(J+1)} \\ &= \frac{2+6-2}{4} = \frac{3}{2}, \\ \frac{\langle S_z \rangle}{\hbar} &= M\frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \\ &= \frac{2+2-6}{4} = -\frac{1}{2}.\end{aligned}$$

The magnetic moment is therefore

$$\begin{aligned}\frac{\mu(^3D_1)}{\mu_N} &= \frac{1}{2}\left(\frac{3}{2}\right) + \frac{g_p + g_n}{2}\left(-\frac{1}{2}\right) \\ &= \frac{3}{4} - \frac{g_p + g_n}{4} = \frac{1}{4}(3 - g_p - g_n).\end{aligned}$$

Numerically,

$$\begin{aligned}3 - g_p - g_n &= 3 - 5.5857 - (-3.8261) \\ &= 3 - 5.5857 + 3.8261 = 1.2404, \\ \therefore \qquad \mu(^3D_1) &= \frac{1.2404}{4}\,\mu_N = 0.3101\,\mu_N.\end{aligned}$$

[6].

Problem 8.6 — Charge independence and no bound nn

Explain why charge independence of the strong force, together with the Pauli principle, is consistent with a bound pn ($s = 1$) deuteron and no bound nn or pp systems.

Solution 8.6

For two nucleons the exchange sign is

$$(-1)^L(-1)^{S+1}(-1)^{T+1} = (-1)^{L+S+T},$$

and Fermi statistics require this sign to be -1. In an S-wave, $L = 0$, so $S + T$ must be odd. The deuteron channel has $S = 1$, hence $T = 0$. Its isospin state

$$\frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle)$$

has no nn or pp member: two identical nucleons necessarily have $T = 1$. For nn or pp with $L = 0$ and $T = 1$, antisymmetry therefore forces $S = 0$. Charge independence relates that channel to the unbound $T = 1$, $^1S_0$ pn channel, apart from electromagnetic and small charge-dependent effects. Thus the bound $T = 0$, $^3S_1$ channel exists only for pn, while the allowed nn and pp S-wave channel does not bind [14, 6].

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