Chapter 1

SEMF and Neutron Stars

"The same short-range force that saturates $B/A \sim 8\,\mathrm{MeV}$ cannot bind pure neutrons; only gravity at stellar mass can."

after Weizsäcker; Bethe (SEMF)

1.1 The puzzle: pure neutrons fail on Earth

The SEMF was built for ordinary nuclei near the valley of stability. If one asks what happens for pure neutron matter, $Z = 0$, the volume and asymmetry terms no longer cancel in a favourable way: the nuclear terms alone leave pure neutrons unbound. Laboratory nuclei never form a stable drop of neutrons.

Nature nevertheless produces compact stellar remnants that are mostly neutrons in the bulk, with masses of order a solar mass and radii of order ten kilometres. Gravity, negligible on the fermi scale of a single nucleus, becomes the leading long-range attraction at $A \sim 10^{56}$–10^{57}. This chapter extrapolates the liquid-drop idea to that extreme and recovers the solar-mass, ten-kilometre scale of neutron stars as an order-of-magnitude estimate [1, 4, 6].

1.1.1 Nuclear landscape and the neutron drip line

Lecture 5 fixed the most stable Z at fixed A. The chart of nuclides (Figure 1.1; cf. Lectures 4–5) also shows, for each Z, a maximum N that still binds. Beyond that **neutron drip line**,

$$S_n(A, Z) = [M(A-1, Z) + m_n - M(A, Z)]c^2 \leq 0, \tag{1.1}$$

and the last neutron is unbound (Lecture 4 separation-energy language). Even the dineutron is unbound; few-neutron systems are unstable. Protons, and the Coulomb–asymmetry balance of Lectures 4–5, are essential for ordinary nuclei.

From the SEMF alone, pure neutron matter ($Z = 0$, $N = A$) has

$$\frac{B}{A} \approx a_V - a_{\mathrm{a}} \approx 15.5 - 23 \approx -7.5\,\mathrm{MeV} < 0 \tag{1.2}$$

(neglecting surface for large A). Asymmetry wins: liquid-drop nuclear physics alone *forbids* a stable pure-neutron drop of any laboratory size.

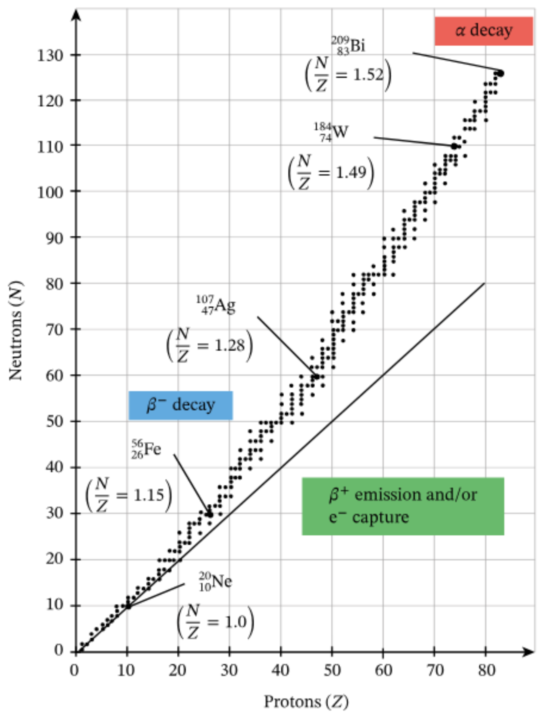

Figure 1.1: Nuclear landscape (valley of stability). For each Z only a limited band of N is bound. Beyond the neutron drip line, $S_n \leq 0$ and neutrons unbind. Laboratory nuclei never reach pure-neutron bulk matter.

1.1.2 Neutron stars exist anyway

Nature produces compact remnants that are mostly neutrons in the bulk (with a thin outer crust of nuclei and a small proton fraction in the core). Observed scales:

Quantity	Typical value
Mass	$\sim (1\text{–}2)\, M_\odot$
Radius	$\sim 10\text{–}15\,\text{km}$
Mean density	$\sim (2\text{–}3)\,\rho_0$ (nuclear saturation)
Nucleon number	$A \sim 10^{57}$ for $M \sim M_\odot$

That is $\sim 10^{55}$ times more nucleons than uranium, in a city-sized volume. The SEMF question: *how can such an object be bound when few neutrons are not?*

1.2 Birth of a neutron star

A compact sketch is enough for this nuclear course [4]:

1. Gas clouds collapse under gravity; fusion begins when the core is hot and dense (H $\to$ He in the Sun).
2. In stars several times heavier than the Sun, fusion proceeds through He, C, O, ... up to the iron-group peak of B/A (Lecture 4). Fusion past iron does not release net energy.

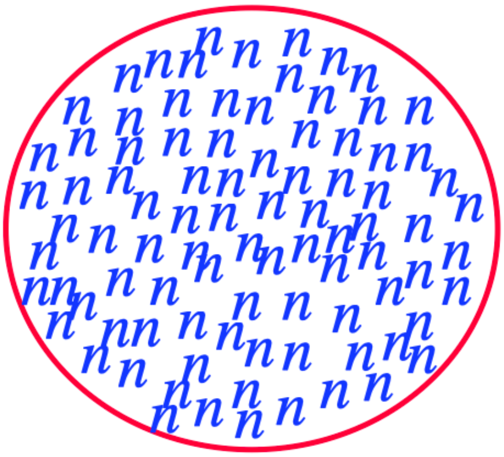

Figure 1.2: Schematic pure-neutron bulk (lecture cartoon). The star is not a giant laboratory nucleus: gravity and quantum degeneracy dominate the global energy and support balance.

3. When nuclear burning can no longer support the core, gravity wins: rapid collapse, bounce, and a core-collapse supernova that ejects the envelope.

4. In the residual core, electron Fermi energies become so high that inverse β decay (electron capture)

$$p + e^- \to n + \nu_e$$

converts most protons into neutrons. Neutrinos escape; the remnant cools into a neutron star.

Why inverse β wins at high density. In free space, $m_n c^2 - m_p c^2 \approx 1.3\,\text{MeV}$ and free neutrons β^- decay. Inside a dense Fermi sea of electrons, capturing an electron from a high Fermi level can lower the total energy of the system even though the free n is heavier. The chemical potentials rearrange until β equilibrium

$$\mu_n = \mu_p + \mu_e \tag{1.3}$$

is reached, typically with a proton fraction of only a few percent in the core: *mostly* neutrons, not purely $Z = 0$.

1.3 Scales: density, number of neutrons, and forces

1.3.1 Number of nucleons

One solar mass is $M_\odot \approx 2 \times 10^{30}\,\text{kg}$. With $m_n \approx 1.67 \times 10^{-27}\,\text{kg}$,

$$A_\odot \approx \frac{M_\odot}{m_n} \approx \frac{2 \times 10^{30}}{1.67 \times 10^{-27}} \approx 1.2 \times 10^{57}. \tag{1.4}$$

A $1.4\,M_\odot$ neutron star therefore contains of order 10^{57} nucleons.

1.3.2 Mean density and the nuclear connection

Nuclear saturation density from Lectures 1–2 is

$$\rho_0 \approx 0.16\,\mathrm{fm}^{-3} \approx 2.7 \times 10^{17}\,\mathrm{kg\,m}^{-3}. \tag{1.5}$$

If a star of mass $M = 1.4\,M_\odot$ has radius $R = 12\,\mathrm{km}$,

$$\bar{\rho} = \frac{3M}{4\pi R^3} \sim (2\text{–}3)\,\rho_0. \tag{1.6}$$

Neutron stars are *nuclear-density objects* writ large: the same saturation that fixed $R = r_0 A^{1/3}$ for Pb reappears as a kilometre-scale radius for $A \sim 10^{57}$.

Radius from nuclear r_0

With $r_0 = 1.2\,\mathrm{fm}$ and $A = 10^{57}$,

$$R = r_0 A^{1/3} = 1.2\,\mathrm{fm} \times 10^{19} = 1.2\,\mathrm{km} \times 10^4 = 12\,\mathrm{km}.$$

Order-of-magnitude agreement with observed radii is not an accident: it is saturation density applied to a stellar baryon number.

1.3.3 Why gravity was dropped in Lectures 4–5

For two nucleons a fermi apart,

$$\frac{Gm_n^2}{r} \sim 10^{-36}\,\mathrm{MeV} \qquad (r \sim 1\,\mathrm{fm}),$$

negligible next to a_V, a_C, or a_a. For $A \sim 10^{57}$ nucleons in a sphere of radius $\sim 10\,\mathrm{km}$, every nucleon feels the gravitational field of the whole star. Total gravitational self-energy scales as $GM^2/R \sim A^{5/3}$ at fixed density and can outweigh a fixed MeV-per-nucleon nuclear deficit.

Key idea

Laboratory nuclei: drop G. Neutron-star scale: keep G. The SEMF structure is reused as energy bookkeeping; gravity is the new long-range term.

1.4 Gravitational self-energy of a uniform sphere

1.4.1 Derivation (same logic as Coulomb 3/5)

Uniform density $\rho = M/(\frac{4}{3}\pi R^3)$. Enclosed mass at radius r:

$$m(r) = M\,\frac{r^3}{R^3}. \tag{1.7}$$

Shell mass:

$$\mathrm{d}m = 4\pi r^2\,\mathrm{d}r\,\rho = \frac{3M}{R^3}\,r^2\,\mathrm{d}r. \tag{1.8}$$

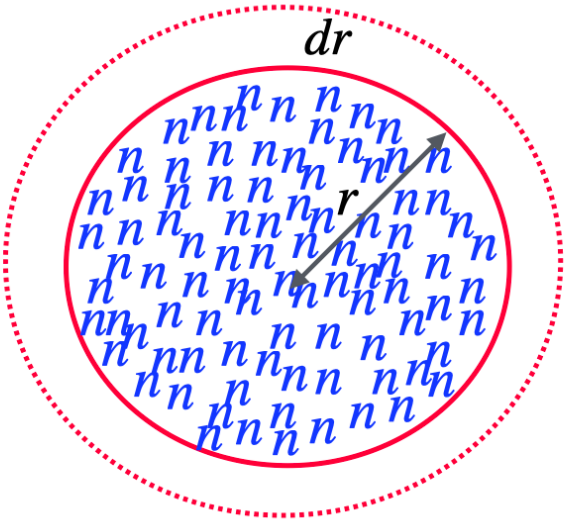

Figure 1.3: Assembly of a uniform sphere by thin shells. A shell of thickness dr at radius r feels the potential of the mass already inside r. Integrating gives $U_G = -3GM^2/(5R)$.

Potential of the interior on the shell:

$$\phi(r) = -\frac{G\,m(r)}{r} = -\frac{GM\,r^2}{R^3}. \tag{1.9}$$

Work to bring dm from infinity:

$$\mathrm{d}U = \phi\,\mathrm{d}m = -\frac{3GM^2}{R^6}\,r^4\,\mathrm{d}r. \tag{1.10}$$

Integrate $0 \to R$:

$$U_G = -\frac{3GM^2}{5R}. \tag{1.11}$$

Compare with the Coulomb self-energy of a uniform charged sphere (Lecture 4):

$$E_\mathrm{C} = \frac{3}{5}\frac{k_e(Ze)^2}{R}, \qquad U_G = -\frac{3}{5}\frac{GM^2}{R}.$$

Same geometric factor 3/5; opposite sign because gravity is attractive.

Binding language. $U_G < 0$ lowers the total energy relative to dispersed matter at infinity. In nuclear $B > 0$ language (Lecture 4), gravity contributes $+|U_G| = 3GM^2/(5R)$ to the binding budget.

Physics checklist for Figure 1.3

1. Shell-by-shell assembly: identical strategy to electrostatics.
2. Scaling: $U_G \propto M^2/R \propto A^{5/3}$ at fixed density.
3. Real stars are stratified; the GM^2/R scaling remains the right order of magnitude.

1.5 SEMF plus gravity for pure neutron matter

1.5.1 Modified binding energy

Start from the SEMF of Lectures 4–5 and add the magnitude of the Newtonian gravitational self-energy as an extra contribution to the *binding* energy (positive when the system is more tightly held together than free nucleons):

$$B = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_\mathrm{a} \frac{(N-Z)^2}{A} + \delta + \frac{3GM^2}{5R}. \tag{1.12}$$

In a total-energy ledger the gravitational self-energy is negative ($U_G = -3GM^2/5R$); writing $+|U_G|$ in B is the same physics packaged as binding. The pure-neutron SEMF exercise below uses this convention only as an order-of-magnitude cartoon.

1.5.2 Idealised pure-neutron limit

Take $Z = 0$, $N = A$, $M = m_n A$, $R = r_0 A^{1/3}$:

- Coulomb vanishes;
- asymmetry: $-a_\mathrm{a}(N-Z)^2/A = -a_\mathrm{a} A$;
- surface $\propto A^{2/3}$ negligible vs volume for huge A (surface/volume $\to 0$; same reason bulk beats surface in macroscopic matter);
- pairing $\propto A^{-3/4}$ negligible at $A \sim 10^{56}$.

Then

$$\frac{3GM^2}{5R} = \frac{3Gm_n^2}{5r_0} A^{5/3}, \tag{1.13}$$

and

$$B \approx (a_V - a_\mathrm{a})\, A + \frac{3Gm_n^2}{5r_0} A^{5/3}. \tag{1.14}$$

Nuclear piece alone is negative.

$$a_V - a_\mathrm{a} \approx -7.5\,\mathrm{MeV}.$$

That is the SEMF version of "you cannot build a stable nucleus from neutrons alone."

Gravity can reverse the sign. The second term grows as $A^{5/3}$. For large enough A it overcomes the 7.5 MeV per-nucleon deficit.

Caution

What this calculation is and is not. It is a pedagogical SEMF extrapolation: an existence argument for a bound pure-neutron system at stellar A. It is *not* modern neutron-star structure. Real support against collapse is neutron **degeneracy pressure** (and nuclear repulsion at short distance) balanced by gravity in general-relativistic hydrostatic equilibrium (Tolman–Oppenheimer–Volkoff equation). Gravity is attractive; degeneracy provides the outward pressure. Here gravity enters only as extra binding in an energy ledger, good for order-of-magnitude A and M, not for precision $M(R)$ curves or the maximum mass.

1.6 Boundedness: critical A, mass, and numerical estimate

Require $B \geq 0$. Divide (1.14) by A:

$$(a_V - a_\mathrm{a}) + \frac{3Gm_n^2}{5r_0} A^{2/3} \geq 0. \tag{1.15}$$

With $a_V - a_\mathrm{a} \approx -7.5\,\mathrm{MeV}$,

$$A^{2/3} \geq \frac{7.5\,\mathrm{MeV}}{C}, \qquad C \equiv \frac{3Gm_n^2}{5r_0}. \tag{1.16}$$

Order of magnitude for C. In SI units,

$$\begin{aligned} G &= 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \\ m_n &= 1.67 \times 10^{-27}\,\mathrm{kg}, \\ r_0 &= 1.2\,\mathrm{fm} = 1.2 \times 10^{-15}\,\mathrm{m}. \end{aligned}$$

Then

$$\begin{aligned} \frac{Gm_n^2}{r_0} &= \frac{6.67 \times 10^{-11}\,(1.67 \times 10^{-27})^2}{1.2 \times 10^{-15}} \approx 1.55 \times 10^{-49}\,\mathrm{J} \\ &\approx 9.7 \times 10^{-37}\,\mathrm{MeV}, \end{aligned}$$

so

$$C = \frac{3}{5}\frac{Gm_n^2}{r_0} \sim 6 \times 10^{-37}\,\mathrm{MeV}.$$

Requiring $CA^{2/3} \gtrsim 7.5\,\mathrm{MeV}$ gives

$$A^{2/3} \gtrsim \frac{7.5}{6 \times 10^{-37}} \sim 1 \times 10^{37}, \qquad A \gtrsim 10^{56}.$$

Thus

$$A \gtrsim 10^{56} \tag{1.17}$$

(lecture estimate; the prefactor depends on the exact gap $a_\mathrm{a} - a_V$ and on r_0).

Mass.

$$M \approx m_n A \sim 10^{56} \times 1.67 \times 10^{-27}\,\mathrm{kg} \sim 1.7 \times 10^{29}\,\mathrm{kg} \sim 0.1\,M_\odot$$

for $A \sim 10^{56}$. Using $A \sim 10^{57}$ (closer to observed $1\,M_\odot$) matches solar-mass neutron stars. The SEMF+gravity argument recovers the *solar-mass order of magnitude*: that is its success [4].

Why $A = 2$, 100, or even 10^{10} still fail

For any laboratory or even planetary A, the gravity term per nucleon is far below 1 eV. Then

$$\frac{B}{A} \approx a_V - a_\text{a} < 0 :$$

unbound. Only stellar A makes $CA^{2/3}$ competitive with 7.5 MeV.

Scale	A	Nuclear B/A	Gravity vs nuclear
Deuteron	2	$\sim +1.1$ MeV (needs $p + n$)	irrelevant
Lead	~ 200	$\sim +8$ MeV	irrelevant
"Neutron drop" (lab)	any small	~ -7.5 MeV if $Z = 0$	still unbound
Neutron star	$\sim 10^{57}$	nuclear + $\lvert U_G \rvert$	gravity decisive

1.7 Physics beyond the SEMF cartoon

1.7.1 What actually supports a neutron star

In hydrostatic equilibrium, gravity pulls inward and a pressure gradient holds the star up. For white dwarfs the pressure is electron degeneracy; for neutron stars it is primarily **neutron degeneracy pressure** (Fermi gas of neutrons at nuclear density), stiffened at high density by repulsive nuclear forces. The SEMF calculation never writes that pressure: it only asks whether the total energy is lower than that of dispersed nucleons. Both viewpoints are useful; they answer different questions (global binding vs local force balance).

1.7.2 Internal structure (qualitative)

- **Atmosphere / outer crust:** ordinary (exotic) nuclei in a lattice with electrons; density rises from $\mathrm{g\,cm^{-3}}$ toward nuclear.
- **Inner crust:** neutron-rich nuclei coexist with a neutron fluid (past the drip density).
- **Outer core:** uniform *npe* matter (neutrons, protons, electrons) in β equilibrium; mostly neutrons.
- **Inner core:** uncertain (hyperons? deconfined quarks?): the regime where laboratory SEMF coefficients are least trustworthy.

1.7.3 Link back to asymmetry and the Fermi gas

The same Pauli filling that produced the asymmetry term $-a_\text{a}(N - Z)^2/A$ (Lecture 4) is the microscopic origin of Fermi kinetic energy. In pure neutron matter the kinetic energy per nucleon is large; the SEMF packages that cost into a_a. Gravity does not cancel the Pauli principle: it adds a different energy scale that grows with system size.

Key idea

SEMF alone: pure neutrons unbound ($a_V < a_\mathrm{a}$). SEMF + gravity: a bound system appears only for $A \gtrsim 10^{56}$, with $M \sim M_\odot$ and $R \sim 10\,\mathrm{km}$ as order-of-magnitude scales. Few neutrons cannot form a star; a stellar number of neutrons can. Realistic stars are not pure neutron and are not held by this energy ledger: they are charge-neutral, β-equilibrated matter supported by an equation of state in hydrostatic (TOV) equilibrium.

1.8 Lessons and course connection

1. Asymmetry energy is real: unbound pure-neutron drops and the need for $N \sim Z$ in light nuclei share one SEMF term.
2. Coefficients a_V, a_a are terrestrial fits; using them at $A = 10^{57}$ is a bold but instructive extrapolation.
3. Gravity changes the ranking of terms: surface, Coulomb, and pairing drop out; volume, asymmetry, and GM^2/R remain.
4. Saturation density links fm and km: $R = r_0 A^{1/3}$ from electron scattering predicts neutron-star radii at the order-of-magnitude level.

Takeaway

Course arc (Lectures 1–6).

- **1–3:** Nuclear size and density (fm, saturation).
- **4–5:** Binding energy, SEMF, most stable Z, mass parabolas.
- **6:** Extreme SEMF: pure neutrons + gravity $\Rightarrow$ neutron-star mass/radius order of magnitude, with pointers to β equilibrium and TOV structure.

Next (Lecture 7). Leave the macroscopic star and return to the microscopic force that underlies a_V: the two-nucleon problem and the deuteron, the only bound NN system.

1.9 Deeper physics: from SEMF cartoon to realistic neutron stars

The pure-neutron SEMF+gravity estimate recovers solar-mass and ten-kilometre scales, but it is not a structure calculation. Three further layers make the cartoon honest without leaving the spirit of this course.

1.9.1 Chemical equilibrium and the proton fraction

In a neutron-star core, weak interactions maintain

$$n \leftrightarrow p + e^- + \bar{\nu}_e \tag{1.18}$$

in equilibrium with charge neutrality $n_p = n_e$ (and muons once $\mu_e > m_\mu c^2$). Chemical equilibrium requires

$$\mu_n = \mu_p + \mu_e. \tag{1.19}$$

Near saturation one may parameterise the energy per nucleon of asymmetric matter as

$$\frac{E}{A} = \varepsilon_0(\rho) + S(\rho)\,(1-2x_p)^2 + \cdots, \tag{1.20}$$

with proton fraction $x_p = n_p/n_b$ and symmetry energy $S(\rho)$. Differentiating and imposing (1.19) with ultra-relativistic electrons ($\mu_e = \hbar c\,(3\pi^2 n_e)^{1/3}$) yields a few-percent x_p near ρ_0: matter is neutron rich but not pure neutron. The SEMF $Z = 0$ exercise is therefore a pedagogical limit, not a literal composition. The asymmetry energy still sets the energy cost of that neutron richness: the same a_a (and its density dependence) that shaped the valley of stability controls the pressure of neutron-star matter [4, 1].

1.9.2 Fermi gas: momentum, energy, and degeneracy pressure

In a volume V the number of neutron spin states with momentum less than p_F is

$$N = 2\cdot\frac{V}{(2\pi\hbar)^3}\cdot\frac{4\pi}{3}p_F^3 = \frac{Vp_F^3}{3\pi^2\hbar^3}. \tag{1.21}$$

Hence

$$p_F = \hbar(3\pi^2 n)^{1/3}, \qquad n = N/V. \tag{1.22}$$

At nuclear saturation density $n \approx \rho_0 \approx 0.16\,\mathrm{fm}^{-3}$,

$$p_F c \approx \hbar c\,(3\pi^2\rho_0)^{1/3} \sim 300\,\mathrm{MeV},$$

so the neutrons are mildly relativistic. In the nonrelativistic limit the kinetic energy density and pressure of a Fermi gas are

$$\varepsilon_\mathrm{kin} = \frac{3}{5}\,n\,\frac{p_F^2}{2m_n}, \tag{1.23}$$

$$P = \frac{2}{3}\,\varepsilon_\mathrm{kin} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_n}\,n^{5/3}. \tag{1.24}$$

That kinetic pressure supports the star against gravity (analogous to electron degeneracy in white dwarfs, but with the neutron mass and nuclear densities). Nuclear potentials modify both energy and pressure: attractive at low density, repulsive at high density. The SEMF volume and asymmetry terms are bulk averages of that physics near ρ_0; they are not a substitute for $P(\varepsilon)$. Pure neutron matter is not self-bound at zero pressure in realistic calculations: stellar gravity and the EOS together make the compact object.

1.9.3 Hydrostatic equilibrium: Newtonian then TOV

Global binding does not replace local force balance. In Newtonian gravity a spherical shell of thickness $\mathrm{d}r$ is in equilibrium when the pressure gradient balances the gravitational force on the enclosed mass $m(r)$:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\,\rho(r)}{r^2}, \qquad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r). \tag{1.25}$$

In general relativity the structure of a static spherical star obeys the Tolman–Oppenheimer–Volkoff (TOV) equations. With energy density $\varepsilon(r)$ (including rest mass) and pressure $P(r)$,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G}{c^2}\frac{(\varepsilon + P)(m + 4\pi r^3 P/c^2)}{r^2(1 - 2Gm/(rc^2))}, \tag{1.26}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\,\frac{\varepsilon}{c^2}. \tag{1.27}$$

Compared with (1.25), three relativistic corrections appear:

1. $\varepsilon + P$ replaces ρc^2 (pressure gravitates);
2. $m + 4\pi r^3 P/c^2$ replaces m (pressure contributes to the enclosed source);
3. the redshift factor $1 - 2Gm/(rc^2)$ in the denominator steepens the gradient near the Schwarzschild scale.

Boundary conditions: regular centre $m(0) = 0$, $P(0) = P_c$; the stellar surface is the radius R where $P(R) = 0$, and the gravitational mass is $M = m(R)$. The input is a cold equation of state $P(\varepsilon)$; the output is a mass–radius curve $M(R)$ and a maximum mass. Observed $M_{\max} \gtrsim 2\,M_\odot$ pulsars require a sufficiently stiff EOS at a few times ρ_0; GW170817 tidal deformability requires that the EOS not be too stiff near $2\rho_0$. The liquid-drop estimate $R \sim 10\,\mathrm{km}$, $M \sim M_\odot$ is the right order of magnitude; precision astrophysics needs the full EOS ladder [4].

Key idea

Three layers of understanding.

1. SEMF + gravity: order-of-magnitude A, M, R (this lecture).
2. Fermi gas + β equilibrium: composition and degeneracy pressure.
3. Realistic EOS + TOV: precision $M(R)$, maximum mass, tides.

This course develops layers 1–2 with honest pointers to layer 3.

1.9.4 Link to laboratory symmetry energy

The slope of the symmetry energy $S(\rho)$ at saturation,

$$L = 3\rho_0 \frac{\mathrm{d}S}{\mathrm{d}\rho}\bigg|_{\rho_0}, \tag{1.28}$$

correlates with the neutron-skin thickness of ^{208}Pb and with neutron-star radii. Lecture 1's charge radii plus parity-violating electron scattering (PREX/CREX) therefore constrain the same physics that sets R_{NS} [1].

1.10 Further physics: nuclear density as the NS density scale

Electron scattering yields a bulk nucleon density

$$\rho_0 \approx 0.15\text{–}0.17\,\mathrm{fm}^{-3} \approx 2.3 \times 10^{14}\,\mathrm{g\,cm}^{-3} \tag{1.29}$$

[14, 1]. A self-gravitating ball of that density with mass $M \sim M_\odot$ has radius

$$R \sim \left(\frac{3M}{4\pi\rho_0}\right)^{1/3} \sim 10\text{–}15\,\mathrm{km}, \tag{1.30}$$

the same order obtained from $R = r_0 A^{1/3}$ with $A \sim 10^{57}$. Thus the laboratory saturation density is the bridge between fermi-scale nuclear physics and kilometre-scale compact stars. Realistic neutron-star models replace the pure-neutron SEMF cartoon by β-equilibrated matter and the TOV equation, but they still pass near ρ_0 in the outer core [14, 4].

1.10.1 Drip lines and pure neutrons

The neutron drip line is where $S_n = 0$. Pure neutron matter ($Z = 0$) lies far beyond that line for any laboratory A: the asymmetry energy costs $\sim a_\mathrm{a} \approx 23\,\mathrm{MeV}$ per nucleon relative to $N = Z$ matter, while volume attraction is only $\sim 16\,\mathrm{MeV}$ net. Gravity at $A \sim 10^{57}$ supplies an extra bulk binding $\propto A^{5/3}/R$ that laboratory nuclei never feel [14, 4].

Compact-star densities still pass near the laboratory saturation density $\rho_0 \sim 0.16\,\mathrm{fm}^{-3}$ in the outer core; that is why electron-scattering radii and neutron-star radii are conceptually linked.

1.10.2 Gravitational scale and Schwarzschild comparison

For $M = M_\odot$ the Schwarzschild radius is

$$R_\mathrm{S} = \frac{2GM}{c^2} \approx 3\,\mathrm{km}. \tag{1.31}$$

Observed neutron-star radii ~ 10–$14\,\mathrm{km}$ are only a few times R_S: general relativity is essential for the structure equations, even though the pedagogical SEMF+gravity estimate of this lecture is Newtonian. White dwarfs, supported by electron degeneracy at much lower density, have $R \sim R_\oplus$ and weaker relativity; the larger critical mass scale for neutron stars reflects the larger Fermi energy of nucleons compared with electrons at white-dwarf densities [14].

1.10.3 Asymmetry energy and the crust

In the outer crust of a neutron star, nuclei coexist with a degenerate electron gas; as density rises, nuclei become more neutron rich until free neutrons drip out ($S_n \to 0$). That threshold is the same drip-line physics as in the laboratory SEMF, now under pressure and in β equilibrium. The bulk asymmetry coefficient a_a and the density dependence of the symmetry energy control the proton fraction in the core and the thickness of the crust: links between nuclear mass fits and compact-star observables [14, 4].

1.10.4 Fermi energy in neutron matter

At $\rho \sim \rho_0$, a pure neutron gas would have a Fermi energy of order tens of MeV (comparable to the laboratory ε_F scale of Lecture 2). β equilibrium $n \leftrightarrow p + e + \bar{\nu}_e$ then fixes a small proton fraction so that $\mu_n = \mu_p + \mu_e$. The SEMF asymmetry term is a schematic way of writing the cost of $N \neq Z$; modern equations of state refine that cost as a function of density, but the order of magnitude is set by the same nuclear saturation physics measured on Earth.

1.11 Problems with solutions

Problem 6.1 — Nucleon number of a solar-mass neutron star

Estimate A for a neutron star of mass $M = 1.4\,M_\odot$ ($M_\odot = 2\times10^{30}$ kg, $m_n = 1.67\times10^{-27}$ kg).

Solution 6.1

$$A \approx \frac{M}{m_n} = \frac{1.4 \times 2 \times 10^{30}}{1.67 \times 10^{-27}} = \frac{2.8 \times 10^{30}}{1.67 \times 10^{-27}} \approx 1.7 \times 10^{57}.$$

Order of magnitude: $A \sim 10^{57}$.

Problem 6.2 — Radius from nuclear density (Petrera 1.1.3)

Using $\rho = 2.3 \times 10^{14}\,\mathrm{g\,cm^{-3}}$ and $M = M_\odot$, recompute R and compare with $R = r_0 A^{1/3}$ for $A = 10^{57}$, $r_0 = 1.2\,\mathrm{fm}$.

Solution 6.2

From density:

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3} = \left(\frac{3 \times 2 \times 10^{33}\,\mathrm{g}}{4\pi \times 2.3 \times 10^{14}\,\mathrm{g\,cm^{-3}}}\right)^{1/3} \approx 13\,\mathrm{km}$$

(cf. Problem 2.2). From $A^{1/3}$:

$$R = 1.2\,\mathrm{fm} \times (10^{57})^{1/3} = 1.2 \times 10^{19}\,\mathrm{fm} = 1.2 \times 10^{4}\,\mathrm{m} = 12\,\mathrm{km}.$$

The two estimates agree: nuclear saturation density sets the kilometre scale of neutron stars.

Problem 6.3 — SEMF pure neutrons unbound

With $a_V = 15.5\,\mathrm{MeV}$ and $a_\mathrm{a} = 23\,\mathrm{MeV}$, show that pure neutron matter ($Z = 0$) has negative binding per nucleon from nuclear terms alone, and estimate the critical A at which $CA^{2/3}$ with $C = 3Gm_n^2/(5r_0) \sim 6 \times 10^{-37}\,\mathrm{MeV}$ can compensate.

Solution 6.3

For $Z = 0$, $N = A$,

$$\frac{B}{A} \approx a_V - a_\mathrm{a} = 15.5 - 23 = -7.5\,\mathrm{MeV}$$

(unbound). Adding the gravity piece $+CA^{2/3}$ per nucleon and setting $CA^{2/3} = 7.5\,\mathrm{MeV}$,

$$A^{2/3} = \frac{7.5}{6 \times 10^{-37}} \sim 1.3 \times 10^{37}, \qquad A \sim (1.3 \times 10^{37})^{3/2} \sim 10^{56}.$$

This is the stellar scale of Lecture 6. (The precise C comes from $3GM^2/(5R)$ with $M = Am_n$, $R = r_0 A^{1/3}$.)

Problem 6.4 — Fermi momentum at saturation

Using $p_F = \hbar(3\pi^2 n)^{1/3}$ and $n = \rho_0 = 0.16\,\mathrm{fm}^{-3}$, compute $p_F c$ in MeV and comment on whether the neutrons are nonrelativistic.

Solution 6.4

$$(3\pi^2 n)^{1/3} = \left(3\pi^2 \times 0.16\right)^{1/3} \approx (4.74)^{1/3} \approx 1.68\,\mathrm{fm}^{-1},$$
$$p_F c = \hbar c \cdot 1.68 \approx 197 \times 1.68 \approx 330\,\mathrm{MeV}.$$

Since $m_n c^2 \approx 940\,\mathrm{MeV}$, one has $p_F c/(m_n c^2) \sim 0.35$: mildly relativistic. The nonrelativistic pressure formula is a first estimate; a realistic EOS must include relativistic kinematics and nuclear interactions.

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