Chapter 1

Most Stable Z and Mass Parabolas

"For constant A, the mass formula represents a parabola of M vs Z."

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1.1 Atomic mass from the SEMF

Lecture 4 constructed the binding energy $B(A, Z)$ and explained how its five terms shape the B/A curve. Stability against β decay is decided not by B alone but by the *atomic* mass $M(A, Z)$: at fixed mass number A, the isobar with the lowest mass cannot lower its energy by converting a neutron into a proton (or vice versa). Writing M as a function of Z at fixed A produces the mass parabolas of this lecture [6, 1, 14].

Key idea

From binding energy to atomic mass.

$$M(A, Z)\, c^2 = [Z\, m(^1\mathrm{H}) + (A - Z)\, m_n]c^2 - B(A, Z).$$

β decay changes Z by ± 1 at fixed A, so it moves along an isobaric slice of this surface. The SEMF turns that slice into a parabola in Z.

Key idea

Which SEMF terms matter at fixed A? When A is held constant, the volume term $a_V A$ and the surface term $a_S A^{2/3}$ are the same for every isobar and drop out of mass *differences*. What remains are:

- the free-nucleon mass piece, linear in Z through $Z\, m(^1\mathrm{H}) + (A - Z)\, m_n$;
- Coulomb, quadratic in Z;
- asymmetry, quadratic in Z through $(A - 2Z)^2$;
- pairing δ, constant on each even/odd Z ladder.

Lecture 4 fixed the bulk physics of B/A; Lecture 5 extracts the Z dependence that controls β stability.

The SEMF binding energy is

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_\mathrm{a} \frac{(N-Z)^2}{A} + \delta(A,Z), \tag{1.1}$$

with $N = A - Z$ and

$$\delta = \begin{cases} +a_P A^{-3/4} & \text{even–even}, \\ 0 & \text{odd } A, \\ -a_P A^{-3/4} & \text{odd–odd}. \end{cases} \tag{1.2}$$

The atomic mass energy is

$$M(A,Z)\, c^2 = \big[Z\, m(^1\mathrm{H}) + (A-Z)\, m_n\big] c^2 - B(A,Z). \tag{1.3}$$

(Using $m(^1\mathrm{H})$ rather than m_p folds the electron mass into the atomic ledger, which is what β energetics require.)

Substituting (1.1) into (1.3) makes the Z dependence explicit:

$$\begin{aligned} M(A,Z)\, c^2 =& \big[Z\, m(^1\mathrm{H}) + (A-Z)\, m_n\big] c^2 \\ & - a_V A + a_S A^{2/3} + a_C \frac{Z(Z-1)}{A^{1/3}} + a_\mathrm{a} \frac{(A-2Z)^2}{A} - \delta. \end{aligned} \tag{1.4}$$

Volume and surface depend only on A. Pairing is constant on each even/odd lattice of Z. The free nucleon (atomic) masses contribute a term linear in Z. Coulomb and asymmetry contribute quadratic pieces in Z.

1.2 Why $M(A,Z)$ is a parabola in Z

Expand the Z-dependent pieces at fixed A. With $N - Z = A - 2Z$,

$$\begin{aligned} a_\mathrm{a} \frac{(A-2Z)^2}{A} &= \frac{a_\mathrm{a}}{A} \big(A^2 - 4AZ + 4Z^2\big) \\ &= a_\mathrm{a} A - 4 a_\mathrm{a} Z + \frac{4a_\mathrm{a}}{A}\, Z^2, \end{aligned} \tag{1.5}$$

and

$$a_C \frac{Z(Z-1)}{A^{1/3}} = \frac{a_C}{A^{1/3}}\, Z^2 - \frac{a_C}{A^{1/3}}\, Z. \tag{1.6}$$

The free-nucleon piece contributes $[m(^1\mathrm{H}) - m_n] c^2\, Z$. Collecting powers of Z,

$$M(A,Z)\, c^2 = \alpha(A) + \beta(A)\, Z + \gamma(A)\, Z^2 - \delta, \tag{1.7}$$

where

$$\beta(A) = \big[m(^1\mathrm{H}) - m_n\big] c^2 - 4 a_\mathrm{a} - \frac{a_C}{A^{1/3}}, \tag{1.8}$$

$$\gamma(A) = \frac{a_C}{A^{1/3}} + \frac{4a_\mathrm{a}}{A} > 0. \tag{1.9}$$

(The linear asymmetry coefficient is $-4a_a$, not $+4a_a/A$: the latter belongs only to the quadratic piece.) The constant $\alpha(A)$ collects volume, surface, the $a_a A$ piece, and the $Am_n c^2$ offset; it does not affect which Z is most stable. Because $\gamma > 0$, $M(Z)$ is a parabola opening *upward*: there is a unique minimum in Z, and nuclei far from that minimum have higher mass and can β decay toward it [6, 1].

Differentiating the quadratic form recovers the same stationarity condition used below: $\partial M/\partial Z = \beta + 2\gamma Z$, and setting that to zero with (1.8)–(1.9) reproduces (1.10).

Geometrically, Coulomb and asymmetry both contribute positive curvature ($\propto Z^2$). Coulomb raises the mass of proton-rich isobars; asymmetry raises the mass when $|N - Z|$ is large. The minimum is the compromise charge where those penalties balance, moderated by the small linear term $\beta(A)Z$ from nucleon mass differences and the linear piece of the Coulomb expansion.

The most stable isobar is the integer Z nearest the minimum of that parabola.

Key idea

At fixed A, Coulomb repulsion and asymmetry energy together make the mass a quadratic in Z. Minimising mass maximises stability against β decay.

1.3 Most stable Z: differentiation of the mass formula

1.3.1 Setting the derivative to zero

Ignore pairing for the moment (exact for odd A; for even A the pairing shifts two parallel parabolas but does not move their common Z_{min} at leading order). Start from (1.4) and differentiate with respect to Z at fixed A. Only the Z-dependent pieces contribute:

$$\frac{\partial}{\partial Z}\left[M(A,Z)\,c^2\right] = \underbrace{\left[m(^1\text{H}) - m_n\right]c^2}_{\text{free-nucleon slope}} + \underbrace{a_C\frac{(2Z-1)}{A^{1/3}}}_{\text{Coulomb slope}} - \underbrace{a_a\frac{4(A-2Z)}{A}}_{\text{asymmetry slope}}. \tag{1.10}$$

The asymmetry derivative uses $\partial(A-2Z)^2/\partial Z = -4(A-2Z)$. Setting the derivative to zero and rearranging so that positive mass-difference and Coulomb terms appear on the right-hand side,

$$\left[m_n - m(^1\text{H})\right]c^2 + a_C\frac{(2Z-1)}{A^{1/3}} = 4a_a\frac{(A-2Z)}{A}. \tag{1.11}$$

The left side favours more protons (neutrons are heavier than hydrogen atoms); the right side favours $N = Z$ through asymmetry. Coulomb enters with the $(2Z-1)$ factor from differentiating $Z(Z-1)$. Collecting terms that multiply Z,

$$Z\left(\frac{2a_C}{A^{1/3}} + \frac{8a_a}{A}\right) = \left[m_n - m(^1\text{H})\right]c^2 + \frac{a_C}{A^{1/3}} + 4a_a, \tag{1.12}$$

and therefore the most stable charge

$$Z_{\text{min}} = \frac{(m_n - m(^1\text{H}))c^2 + a_C A^{-1/3} + 4a_a}{2a_C A^{-1/3} + 8a_a/A}. \tag{1.13}$$

This is the exact stationary condition for the SEMF mass parabola [6, 1]. The most stable nuclide is the integer Z closest to $Z_{\min}$.

1.3.2 Numerical size of the free-mass term

From atomic mass units,

$$m_n - m(^1\mathrm{H}) = 1.008665 - 1.007825 = 0.000840\,\mathrm{u}, \tag{1.14}$$

so

$$[m_n - m(^1\mathrm{H})]c^2 \approx 0.000840 \times 931.5 \approx 0.782\,\mathrm{MeV}. \tag{1.15}$$

With the usual coefficients $a_C = 0.72\,\mathrm{MeV}$ and $a_\mathrm{a} = 23\,\mathrm{MeV}$, for $A \sim 100$ one has

$$\frac{a_C}{A^{1/3}} \approx 0.16\,\mathrm{MeV}, \qquad \frac{4a_\mathrm{a}}{A} \approx 0.92\,\mathrm{MeV},$$

while $4a_\mathrm{a} = 92\,\mathrm{MeV}$ dominates the numerator of (1.13). The free-mass difference is not negligible next to $a_C/A^{1/3}$, but both are small compared with the asymmetry scale [6].

1.3.3 Working formula used in the lecture

Neglecting the small free-mass and linear-Coulomb pieces relative to $4a_\mathrm{a}$ in the numerator, and writing

$$Z_{\min} \approx \frac{4a_\mathrm{a}}{2a_C A^{-1/3} + 8a_\mathrm{a}/A} = \frac{A/2}{1 + \frac{1}{4}(a_C/a_\mathrm{a})\,A^{2/3}}, \tag{1.16}$$

one obtains with $a_C/a_\mathrm{a} = 0.72/23 \approx 0.0313$

$$Z_{\min} \approx \frac{A}{2(1 + 0.0078\,A^{2/3})}. \tag{1.17}$$

A closely related fit uses 0.0075 instead of 0.0078 in the denominator [14].

Key idea

Light vs heavy nuclei. For small A, $0.0078\,A^{2/3} \ll 1$, so $Z_{\min} \approx A/2$ ($N \approx Z$): oxygen, carbon, calcium. For large A the Coulomb correction grows, $Z_{\min} < A/2$, and long-lived nuclei become neutron rich ($N/Z \sim 1.5$ for lead). Equation (1.17) gives $Z/A \approx 0.41$ for heavy species, matching observation [6, 1].

A	$Z_{\min}$ (approx.)	Nearest stable Z	N/Z
16	7.6	8 (^{16}O)	1.0
40	18.3	20 (^{40}Ca)	1.0
56	25.1	26 (^{56}Fe)	1.2
125	52.3	52 (^{125}Te)	1.4
208	81.6	82 (^{208}Pb)	1.5

Values of $Z_{\min}$ come from (1.17); the nearest integer stable Z can differ by one when the parabola minimum falls between integers.

1.4 Odd A: one parabola and β decay toward the minimum

For odd A, $\delta = 0$. There is a *single* mass parabola. Nuclei with Z far from $Z_{\min}$ have large mass differences to their neighbours and decay by $\beta^{\pm}$ or electron capture; nuclei near the minimum are stable or long-lived.

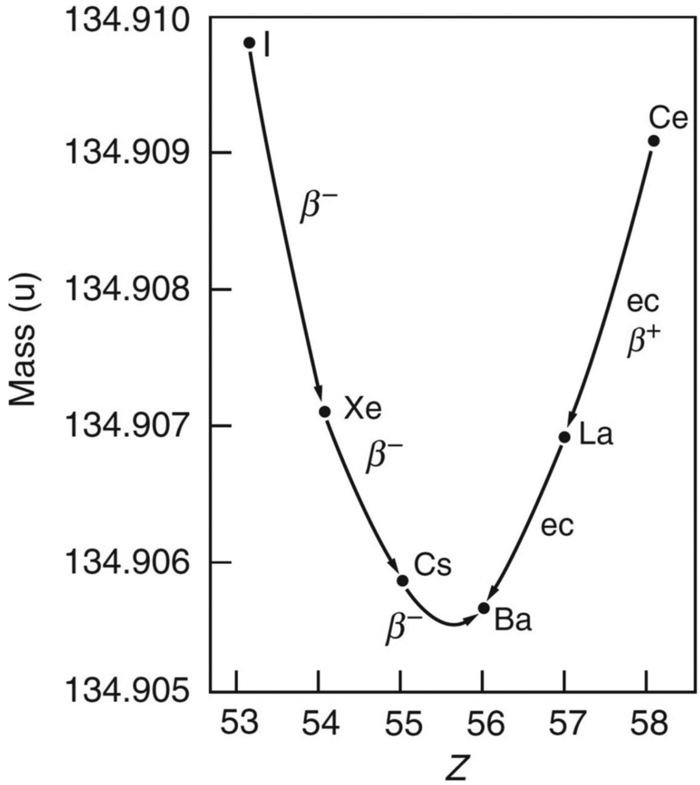

Figure 1.1: Mass parabola for odd $A = 135$ (lecture figure). One minimum near $Z = 56$; one (or few) β-stable isobar(s). Decay energy increases as one moves away from the minimum [1, 6].

1.4.1 β energetics on the parabola

Along an odd-A isobar the available energy for a single β step is the atomic mass difference of neighbouring isobars. For β^- ($Z \to Z + 1$),

$$Q_{\beta^-} = \big[M(A, Z) - M(A, Z + 1)\big]c^2. \tag{1.18}$$

For β^+ ($Z \to Z - 1$),

$$Q_{\beta^+} = \big[M(A, Z) - M(A, Z - 1)\big]c^2 - 2m_e c^2 \tag{1.19}$$

(the extra $2m_e c^2$ appears when atomic masses are used carefully for positron emission). Electron capture has a slightly lower threshold than β^+.

Because $M(Z)$ is parabolic, the mass difference between neighbours *grows* as one moves away from $Z_{\min}$. That is why β-decay Q values (and typically the decay rates) increase as one climbs the sides of the parabola [6, 1].

Key idea

Direction of β decay from geometry alone.

$Z < Z_{\min}$	neutron rich	β^- $(n \to p + e^- + \bar{\nu}_e)$,
$Z > Z_{\min}$	proton rich	β^+ or electron capture,
$Z \approx Z_{\min}$	stable / long-lived	no allowed single β.

The SEMF fixes the *direction* and approximate Q; Fermi theory fixes spectra and lifetimes.

1.5 Even A: pairing splits the mass surface into two parabolas

For even A, the pairing term shifts even–even nuclei *down* and odd–odd nuclei *up* by $|\delta| = a_P A^{-3/4}$ in the binding-energy convention of (1.1). Because $M\,c^2$ contains $-\delta$, the mass of an even–even isobar is lowered by $|\delta|$ and that of an odd–odd isobar is raised by $|\delta|$. The vertical separation between the two parabolas is therefore $2|\delta|$. The mass surface becomes two parallel parabolas sharing the same $Z_{\min}$ but offset in energy [6, 1, 14].

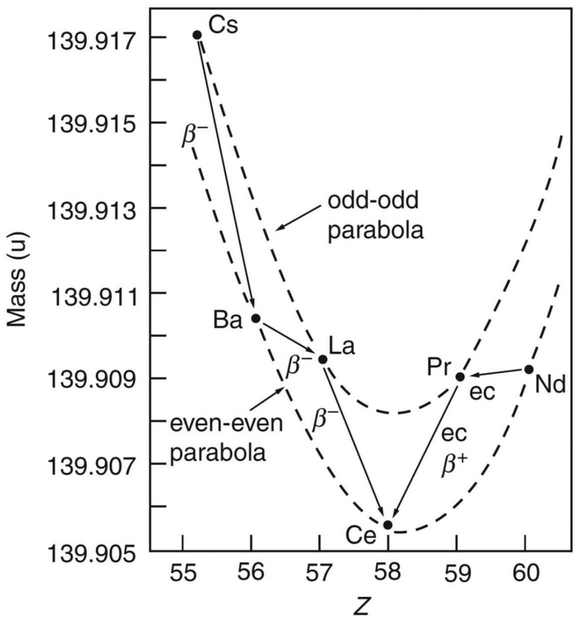

Figure 1.2: Even $A = 140$: pairing splits even–even (lower) and odd–odd (upper) chains. One β-stable even–even isobar near $Z = 58$.

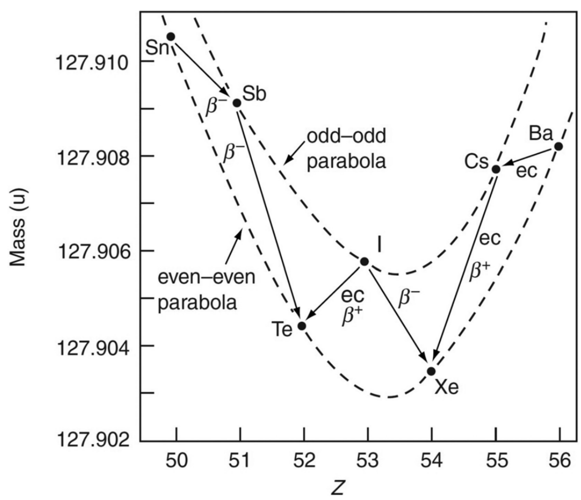

Figure 1.3: Even $A = 128$: two even–even isobars ($Z = 52, 54$) lie on the lower parabola; the odd–odd member can decay both ways. Some even–even chains are stable against *single* β but decay very slowly by double β (e.g. tellurium isotopes).

1.5.1 Two effects absent from odd-A chains

1. **Bidirectional β decay of odd–odd nuclei.** An odd–odd isobar near the minimum of the *upper* parabola can have both neighbours on the lower parabola at lower mass. Then both

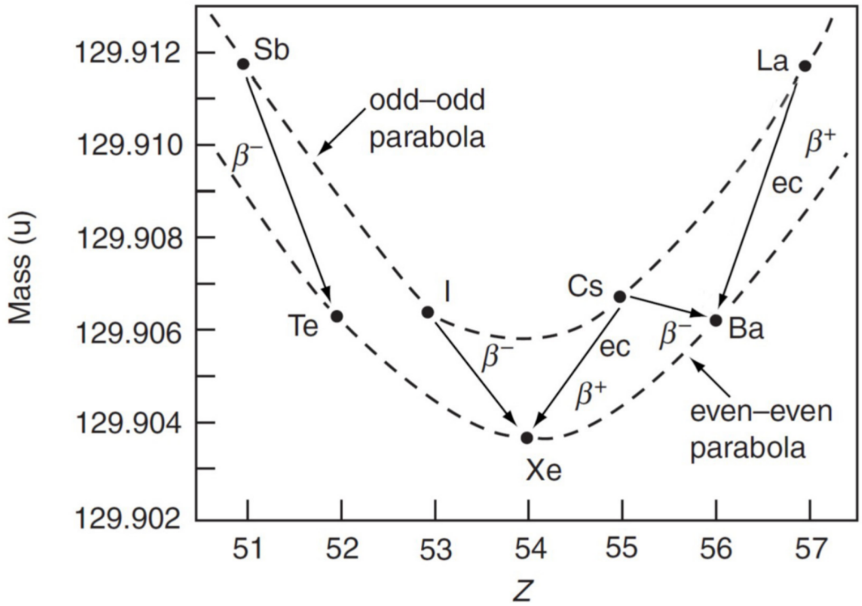

Figure 1.4: Even $A = 130$: three even–even nuclides ($Z = 52, 54, 56$) on the lower parabola can all be stable against single β decay relative to their odd–odd neighbours. Long-term evolution may still proceed by double β where $Q_{2\beta} > 0$.

β^- and β^+/EC are open. Classic example: ^{128}I can convert either a neutron or a proton to reach a more bound even–even neighbour [6, 1].

2. **Double β decay of even–even nuclei.** If single β from an even–even nucleus would land on a higher odd–odd isobar, that single step is energetically forbidden, yet a transition to a lower even–even neighbour two units of Z away can still be allowed. The nucleus may then decay only by **double beta decay** ($2\nu\beta\beta$, or the still-unseen $0\nu\beta\beta$). Candidates include ^{76}Ge, ^{128}Te, ^{130}Te, and ^{136}Xe, where single β is blocked by pairing but $Q_{2\beta} > 0$ [6, 1, 4]. Observationally, such nuclei can be extremely long lived: ^{128}Te has a $2\nu\beta\beta$ half-life of order 10^{24} yr.

Key idea

Pairing does not invent new forces; it rearranges which isobars are allowed to β decay. Odd A: one parabola, one valley. Even A: two parabolas, possible double-β and two-way decays.

1.5.2 Why light odd–odd nuclei can be stable

For small A the parabola is narrow (the coefficient $\gamma(A)$ is larger). The pairing offset can then place an odd–odd nucleus low enough to be stable against single β. The four stable odd–odd species ^{2}H, ^{6}Li, ^{10}B, ^{14}N live in that regime; heavier odd–odd nuclei almost always find an open β path to an even–even daughter [1, 6].

1.6 β Q values from the SEMF (worked structure)

Using atomic masses, the β^- energy release between neighbouring isobars is

$$Q_{\beta^-}(A,Z) = [M(A,Z) - M(A,Z+1)]c^2 = [m_n - m(^1\mathrm{H})]c^2 + B(A,Z+1) - B(A,Z) \quad (1.20)$$

(up to the usual atomic-mass bookkeeping). Substituting the SEMF, volume and surface cancel at fixed A, and one is left with Coulomb, asymmetry, and pairing differences, exactly the terms that shape the parabola [3, 14].

Similarly for β^+,

$$Q_{\beta^+}(A,Z) = [M(A,Z) - M(A,Z-1)]c^2 - 2m_e c^2, \quad (1.21)$$

so both Coulomb (fewer protons helps) and asymmetry (moving toward $N = Z$ or away from it) enter with opposite roles on the two sides of the valley.

Worked estimate: Q_{β^-} for $^{135}\mathrm{Cs}$

For odd $A = 135$, pairing vanishes and the SEMF gives a single parabola. With $a_C = 0.72$, $a_\mathrm{a} = 23$ (MeV), $Z_\mathrm{min} \approx 56$ so the valley sits near barium ($Z = 56$). Consider neutron-rich $^{135}\mathrm{Cs}$ ($Z = 55$):

$$Q_{\beta^-} = [M(135,55) - M(135,56)]c^2.$$

Volume and surface cancel at fixed A. Using atomic masses in the SEMF,

$$Q_{\beta^-} \approx [m_n - m(^1\mathrm{H})]c^2 + a_C \frac{56\cdot 55 - 55\cdot 54}{135^{1/3}} + a_\mathrm{a} \frac{(135-110)^2 - (135-112)^2}{135}.$$

The free-mass piece is $+0.78\,\mathrm{MeV}$. The Coulomb piece is

$$0.72 \cdot \frac{56}{135^{1/3}} \approx 0.72 \cdot \frac{56}{5.13} \approx 7.9\,\mathrm{MeV}$$

(increasing Z costs Coulomb energy, which *reduces* Q when written as a mass difference favoring the daughter—equivalently the parent has smaller Coulomb energy). More carefully, the contribution to $M(Z) - M(Z+1)$ from Coulomb is

$$a_C A^{-1/3}[Z(Z-1) - (Z+1)Z] = -a_C A^{-1/3}(2Z) = -0.72 \cdot \frac{110}{5.13} \approx -15.4\,\mathrm{MeV}.$$

The asymmetry contribution to $M(Z) - M(Z+1)$ is

$$a_\mathrm{a} A^{-1}[(A-2Z)^2 - (A-2Z-2)^2] = a_\mathrm{a} A^{-1}[4(A-2Z) - 4] \approx 23 \cdot \frac{4\cdot 25 - 4}{135} \approx 16.6\,\mathrm{MeV}.$$

Altogether

$$Q_{\beta^-} \approx 0.78 - 15.4 + 16.6 \approx 2.0\,\mathrm{MeV}.$$

The measured value is $Q \sim 1.3\,\mathrm{MeV}$: the SEMF captures the direction and the MeV scale, not every keV [6, 1]. The positive sign confirms β^- decay of neutron-rich ^{135}Cs toward ^{135}Ba.

Physics checklist for an isobar

1. Compute $Z_{\min}(A)$ from (1.17) (or (1.13)).
2. For odd A: one parabola; the integer Z nearest $Z_{\min}$ is stable; others β decay toward it with Q increasing up the sides.
3. For even A: draw two parabolas split by $\sim 2|\delta|$; identify even–even minima and any odd–odd bidirectional cases; mark possible double-β pairs.
4. Q_β follows from mass differences on that surface, with no weak matrix element required for the *sign* of the decay.

1.7 Connecting the valley of stability

Plotting the integer nearest to $Z_{\min}(A)$ for each A traces the valley of stability on the (N, Z) plane: $N \approx Z$ at low A, bending to $N > Z$ at high A. Lectures 4–5 therefore close the liquid-drop programme: binding energy and its Z landscape, including the β arrows that keep real nuclei near the valley [6, 1, 13].

1.8 Deeper physics: exact $Z_{\min}$ and isobar energetics

1.8.1 Complete mass formula in nuclear masses

The same parabola follows from writing the atomic mass with nuclear constituents,

$$M(A,Z) = (A-Z)m_n + Z(m_p+m_e) - \frac{a_V A}{c^2} + \frac{a_S A^{2/3}}{c^2} + \frac{a_C Z(Z-1)}{c^2 A^{1/3}} + \frac{a_\mathrm{a}(A-2Z)^2}{c^2 A} - \frac{\delta}{c^2}, \tag{1.22}$$

which differs from (1.4) only by the usual $m(^1\mathrm{H})$ versus $m_p + m_e$ bookkeeping. Setting $\partial M/\partial Z = 0$ yields

$$Z_{\min} = \frac{(m_n - m_p - m_e)c^2 + a_C A^{-1/3} + 4a_\mathrm{a}}{2a_C A^{-1/3} + 8a_\mathrm{a}/A}, \tag{1.23}$$

equivalent to (1.13). The free-mass difference $(m_n - m_p - m_e)c^2 \approx 0.78\,\mathrm{MeV}$ is small next to $4a_\mathrm{a} \sim 90\,\mathrm{MeV}$; dropping it and the linear Coulomb piece recovers (1.17) [1].

Why heavy nuclei need extra neutrons

Coulomb pushes $Z_{\min}$ below $A/2$. Each missing proton relative to $A/2$ is replaced by a neutron, so $N - Z = A - 2Z_{\min}$ grows with A. The asymmetry term fights that growth; equilibrium is the compromise in (1.23). Without Coulomb, $Z_{\min} = A/2$ exactly (up to the tiny $m_n - m_p$ shift).

1.8.2 Parabola accuracy and Q_β residuals

At fixed A, the SEMF reproduces mass *differences* between neighbouring isobars to within a few hundred keV for medium-mass nuclei once the overall offset is fixed by the lowest isobar. Shell corrections and pairing details appear as residuals of order 0.5–1 MeV [14, 6]. Near $Z_{\min}$ the Q_β values are therefore small; far up the parabola they grow, setting the scale for Fermi-theory spectra in a later chapter.

1.9 Further physics: atomic minimum and chart systematics

1.9.1 Atomic mass minimum and the $0.78\,\mathrm{MeV}$ shift

Stability against β decay is decided by the *atomic* mass, not the bare nuclear mass. Minimising $M_{\text{atom}}(A, Z)$ includes the neutron–hydrogen mass difference $\delta_{nH} = (m_n - m(^1\mathrm{H}))c^2 \approx 0.78\,\mathrm{MeV}$ in $Z_{\min}$. Light nuclei slightly favour protons relative to a pure $N = Z$ Fermi-gas estimate, while Coulomb still forces $N > Z$ for heavy systems [14, 6].

1.9.2 Odd A one parabola; even A two

For odd A (pairing zero) masses follow one parabola in Z. For even A, even–even and odd–odd species form two parabolas split by $\sim 2|\delta|$. Double-β candidates sit on the even–even curve when single β would land on the higher odd–odd curve [14, 5, 13, 1].

A useful illustration is $A = 112$: ^{112}Sn and ^{112}Cd can both be stable against single β on the lower even–even parabola, while intermediate odd–odd ^{112}In lies higher and can decay both ways [14].

1.9.3 Systematics on the chart of nuclides

On the (N, Z) plane:

- nuclei *below* the valley (neutron excess) decay by β^-;
- nuclei *above* the valley (proton excess) decay by β^+ or electron capture;
- the valley bends toward $N > Z$ as A grows, as $Z_{\min}(A) < A/2$.

Plotting the integer nearest to $Z_{\min}(A)$ reproduces the valley curve seen on the chart of nuclides in Lecture 4.

1.10 Problems with solutions

Problem 5.1 — Evaluate $Z_{\min}$

Using $Z_{\min} = A/\left[2(1 + 0.0078\,A^{2/3})\right]$, compute $Z_{\min}$ for $A = 100$ and $A = 200$. For each, give N/Z at that charge.

Solution 5.1

For $A = 100$, $A^{2/3} \approx 21.54$,

$$Z_{\min} = \frac{50}{1 + 0.0078 \times 21.54} = \frac{50}{1.168} \approx 42.8 \quad \Rightarrow \quad \frac{N}{Z} \approx \frac{57.2}{42.8} \approx 1.34.$$

For $A = 200$, $A^{2/3} \approx 34.2$,

$$Z_{\min} = \frac{100}{1 + 0.0078 \times 34.2} = \frac{100}{1.267} \approx 78.9 \quad \Rightarrow \quad \frac{N}{Z} \approx \frac{121}{79} \approx 1.53.$$

Heavy nuclei are neutron rich [6].

Problem 5.2 — Free-mass difference

Show that $(m_n - m(^1\mathrm{H}))c^2 \approx 0.782\,\mathrm{MeV}$ using $m_n = 1.008665\,\mathrm{u}$, $m(^1\mathrm{H}) = 1.007825\,\mathrm{u}$, and $c^2 = 931.5\,\mathrm{MeV/u}$. Where does this number enter $Z_{\min}$?

Solution 5.2

$$(1.008665 - 1.007825) \times 931.5 = 0.000840 \times 931.5 \approx 0.782\,\mathrm{MeV}.$$

It appears in the numerator of the exact $Z_{\min}$ formula (1.13) as the free-mass drive toward protons; it is small compared with $4a_{\mathrm{a}} \approx 92\,\mathrm{MeV}$ [6, 14].

Problem 5.3 — Q_{β^-} sign on the parabola

At fixed odd A, $M(Z)$ is a parabola with minimum at $Z_{\min}$. Show that $Q_{\beta^-} = [M(Z) - M(Z+1)]c^2 > 0$ for all integers $Z < Z_{\min} - 1/2$, so neutron-rich isobars β^- decay toward the minimum.

Solution 5.3

Write $M(Z) = \alpha - \beta Z + \gamma Z^2$ with $\gamma > 0$. Then

$$M(Z) - M(Z+1) = \beta - \gamma(2Z + 1).$$

The minimum of M is at $Z_{\min} = \beta/(2\gamma)$, so

$$M(Z) - M(Z+1) = 2\gamma(Z_{\min} - Z - \tfrac{1}{2}).$$

For $Z \le Z_{\min} - 1$ (integer steps below the vertex), this difference is positive: $Q_{\beta^-} > 0$. Geometry alone forces β^- on the neutron-rich side [6, 1].

Problem 5.4 — Pairing split and double β

Even–even and odd–odd mass parabolas at fixed even A are split by $2|\delta|$ with $|\delta| = a_P A^{-3/4}$. For $A = 128$ and $a_P = 34\,\mathrm{MeV}$, estimate the vertical split $2|\delta|$. Explain in one sentence why an even–even nucleus can be stable against single β yet unstable against double β.

Solution 5.4

$$A^{3/4} = 128^{0.75} \approx (2^7)^{0.75} = 2^{5.25} \approx 38, \quad |\delta| \approx 34/38 \approx 0.9\,\mathrm{MeV}, \quad 2|\delta| \approx 1.8\,\mathrm{MeV}.$$

Single β would land on the higher odd–odd parabola, so $Q_\beta < 0$ for that step, while a jump of $\Delta Z = 2$ to a lower even–even neighbour can still have $Q_{2\beta} > 0$ [6, 1].

Problem 5.5 — Both β modes (Petrera-style)

For odd–odd ^{64}Cu near mid-mass, explain qualitatively why both β^- (to ^{64}Zn) and β^+ (to ^{64}Ni) can be open, using the two-parabola picture.

Solution 5.5

^{64}Cu sits on the upper (odd–odd) parabola. Both even–even neighbours ^{64}Ni and ^{64}Zn lie on the lower parabola and can be lower in mass than ^{64}Cu. Then both Q_{β^-} and Q_{β^+} are positive: bidirectional decay. Quantitative SEMF estimates give $Q \sim 1\,\mathrm{MeV}$ for both branches [3, 6].

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