Chapter 1

Mass Density and Saturation

"The central nuclear charge density is nearly the same for all nuclei. Nucleons do not congregate near the center."

K. S. Krane

1.1 From charge density to mass density

Electron scattering maps the proton (charge) density $\rho_p(r)$. A central empirical fact is that this density is nearly the same in the interior of all nuclei: nucleons do not pile up at the centre. Empirically one fits

$$\rho_p(r) = \frac{\rho_p(0)}{1 + \exp\big[(r - R)/a\big]}, \tag{1.1}$$

with a nearly universal surface diffuseness $a = 0.52\,\mathrm{fm} \pm 0.01\,\mathrm{fm}$ and a half-density radius $R_0 = 1.2\,\mathrm{fm} \times A^{1/3}$ [1, 6].

Electrons do not couple to neutrons. The full nucleon density, often called the mass or matter density, must be reconstructed from the charge density together with a model of how neutrons are distributed, or measured with hadronic probes. That matter density, and the near constancy of its central value, is the subject of this chapter. It is also the microscopic reason the strong force must be short ranged: if every nucleon attracted every other nucleon, neither the density nor the binding energy would scale as they do [6].

Here $\rho_p(r)$ denotes the *proton number density* (nucleons per fm^3), not the electromagnetic charge density $e\,\rho_p$. Electron scattering determines ρ_p because the charge density is $e\,\rho_p$ for point-like protons (up to small meson-exchange corrections). It is equally important to know the distribution of *all* nucleons (neutrons and protons), because nuclear binding, the equation of state, and the liquid-drop model all refer to the total nucleon number density $\rho(r)$. (The true mass density is $m_N\rho$; nuclear physics usually quotes number densities.)

Since neutrons are uncharged, ordinary Coulomb scattering does not observe them. Under a standard working assumption, however, one can still infer $\rho(r)$ from $\rho_p(r)$.

1.1.1 The constant N/Z assumption

A nucleus with N neutrons and Z protons has overall ratio N/Z and mass number $A = N + Z$. It is usually assumed that the local neutron-to- proton density ratio is the same everywhere

inside the nucleus:

$$\frac{\rho_n(r)}{\rho_p(r)} = \frac{N}{Z}. \tag{1.2}$$

The total nucleon density is the sum

$$\rho(r) = \rho_n(r) + \rho_p(r). \tag{1.3}$$

Eliminating ρ_n between these two equations gives immediately

$$\rho(r) = \rho_p(r)\left(1 + \frac{N}{Z}\right) = \rho_p(r)\,\frac{A}{Z}. \tag{1.4}$$

Definition: nucleon (mass) density from charge density

Under the assumption $\rho_n/\rho_p = N/Z$ everywhere,

$$\rho(r) = \frac{A}{Z}\,\rho_p(r).$$

Rescaling the measured proton density by A/Z converts a charge distribution into an estimate of the total nucleon density.

Caution

Equation (1.2) is an approximation, not a theorem. In heavy nuclei ($N > Z$) a *neutron skin* develops: $\rho_n(r)$ extends farther than $\rho_p(r)$. The constant-ratio assumption then fails in the surface. For the bulk and for light or near-symmetric nuclei it remains a useful first estimate; for precision neutron-skin physics one needs parity-violating electron scattering or hadronic probes (cf. Lecture 1).

1.1.2 What the rescaled densities look like

Figure 1.1 recalls the proton densities of ^{16}O, ^{118}Sn, and ^{197}Au. Applying Equation (1.4) produces the nucleon densities of Figure 1.2. Two features stand out at once.

1. The half-density radii and surface widths are essentially the same as in the charge densities: rescaling multiplies the vertical axis, not the radial shape.
2. All three nuclei now share virtually the *same central nucleon density*, $\rho_0 \approx 0.16 - -0.17\,1/\mathrm{fm}^3$.

Key idea

Saturation of nuclear matter: the central nucleon density is nearly independent of mass number [1]. Nuclei are not denser in the middle when they grow larger; they grow by adding a larger volume at fixed bulk density. This confirms the strong interaction is short ranged and dominated by nearest-neighbour nucleons.

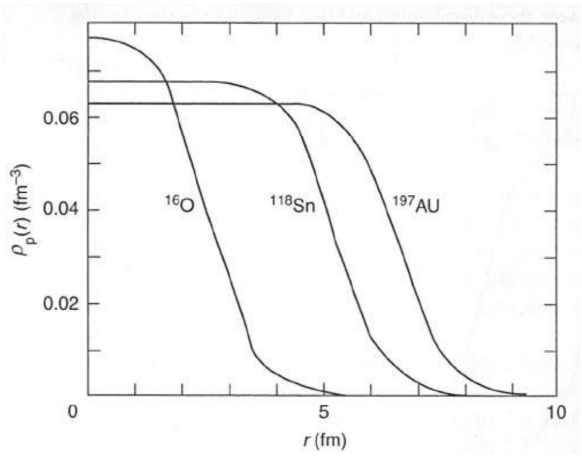

Figure 1.1: Proton densities $\rho_p(r)$ for ^{16}O, ^{118}Sn, and ^{197}Au.

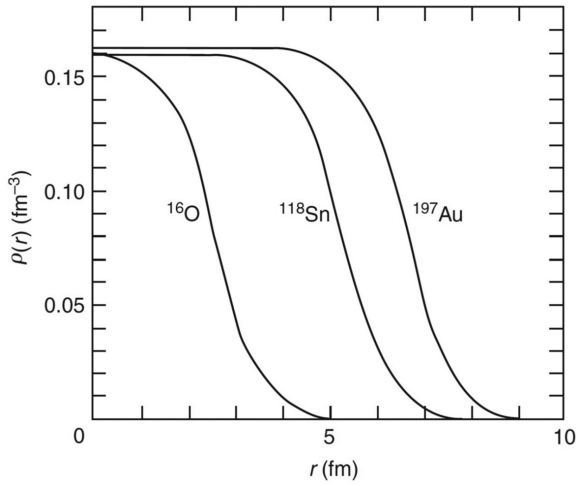

Figure 1.2: Nucleon densities $\rho = (A/Z)\rho_p$ with common central ρ_0.

1.2 Woods–Saxon parameters and the nuclear surface

The nucleon density is again described by a Woods–Saxon (two-parameter Fermi) profile,

$$\rho(r) = \frac{\rho_0}{1 + \exp\big[(r - R)/a\big]}, \tag{1.5}$$

where now ρ_0 is the central *nucleon* density, R is the half-density radius, and a is the diffuseness.

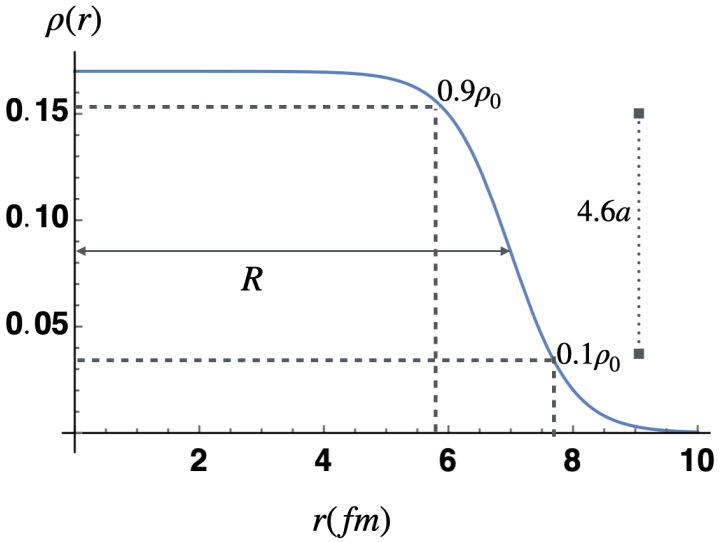

Figure 1.3: Geometry of the Woods–Saxon surface. The half-density radius R is the point where $\rho = \rho_0/2$. The surface thickness t, from $0.9\rho_0$ to $0.1\rho_0$, equals $4a \ln 3 \approx 4.4\,a$.

1.2.1 Surface thickness from first principles

To relate a to an observable surface thickness, evaluate the radii at which the density has fallen to 90% and 10% of ρ_0.

Setting $\rho = 0.9\rho_0$,

$$0.9\rho_0 = \frac{\rho_0}{1+e^{(r-R)/a}} \implies e^{(r-R)/a} = \frac{1}{9} \implies \frac{r-R}{a} = -\ln 9 = -2\ln 3. \tag{1.6}$$

Hence

$$r_{90} = R - a\ln 9 = R - 2a\ln 3 \approx R - 2.20\,a. \tag{1.7}$$

Setting $\rho = 0.1\rho_0$,

$$0.1\rho_0 = \frac{\rho_0}{1+e^{(r-R)/a}} \implies e^{(r-R)/a} = 9 \implies r_{10} = R + 2a\ln 3 \approx R + 2.20\,a. \tag{1.8}$$

The surface thickness is therefore

$$t \equiv r_{10} - r_{90} = 4a\ln 3 \approx 4.39\,a. \tag{1.9}$$

Remark

Lecture notes sometimes quote $t \approx 4.6a$ by approximating $\ln 9 \approx 2.3$. The exact factor is $4\ln 3 \approx 4.394$. A standard global fit gives $a = 0.52\,\text{fm} \pm 0.01\,\text{fm}$ [1], so

$$t = 4a\ln 3 \approx 4.39 \times 0.52\,\text{fm} \approx 2.28\,\text{fm}.$$

The nuclear "skin" is only a couple of fermis thick and is nearly independent of A. (The sketch in Figure 1.3 labels the thickness $4.6a$; we keep the exact $4\ln 3$.)

Example: surface of a heavy nucleus

With $a = 0.52\,\text{fm}$,

$$t \approx 2.3\,\text{fm}.$$

For ^{197}Au, $R_0 = 1.2\,\text{fm} \times 197^{1/3} \approx 7.0\,\text{fm}$ [1]. The density is essentially flat for $r \lesssim 5\,\text{fm}$ and has fallen to a few percent of ρ_0 by $r \approx 9 - -10\,\text{fm}$, matching Figure 1.2.

The central core is uniformly populated with nucleons up to $r \sim R$; beyond that the density decreases smoothly. The nucleus does not have a sharp boundary but a diffuse quantum surface.

1.2.2 The $A^{1/3}$ law for the half-density radius

The mean nuclear radius R_0 increases with mass number. Saturation already implies the functional form. If the central density is A-independent, the volume that holds A nucleons must grow as

$$A = \tfrac{4}{3}\pi R^3\,\rho_0 \quad \Rightarrow \quad R = \left(\frac{3}{4\pi\rho_0}\right)^{1/3} A^{1/3} \equiv r_0\,A^{1/3}. \tag{1.10}$$

With $\rho_0 \approx 0.16\,\text{fm}^{-3}$,

$$r_0 = \left(\frac{3}{4\pi \times 0.16}\right)^{1/3} \approx 1.14\,\text{fm}, \tag{1.11}$$

close to the conventional fit value. Analysis of electron-scattering data yields the empirical expression [1]

$$R_0 = 1.2\,\text{fm} \times A^{1/3}. \tag{1.12}$$

Combined with a constant central density, this is precisely the geometry of an incompressible liquid drop: volume $\propto A$, so radius $\propto A^{1/3}$. Equation (1.12) is a standard input to the liquid-drop model [1].

The mean inter-nucleon spacing at saturation follows from the same density:

$$d \sim \rho_0^{-1/3} \approx (0.16)^{-1/3}\,\text{fm} \approx 1.8\,\text{fm}, \tag{1.13}$$

comparable to the pion Compton wavelength that will appear in Lecture 11.

Takeaway

Three numbers characterise bulk nuclear matter as seen by electrons (rescaled to nucleons) [1]:

$$\rho_0 \approx 0.16/\text{fm}^3, \quad R_0 = 1.2\,\text{fm}\,A^{1/3}, \quad a = 0.52\,\text{fm} \pm 0.01\,\text{fm} \;\; (t \approx 2.3\,\text{fm}).$$

They will reappear in the liquid-drop model, the semi-empirical mass formula, and the nuclear equation of state.

1.3 What uniform density teaches about the strong force

The nucleus exists because a strong attractive force counters the Coulomb repulsion among protons. Without that force the nucleus would explode. Nuclear attraction is far stronger than Coulomb repulsion at fermi distances, yet the density profile is *flat*, not centrally peaked.

One might have expected the opposite. In a self-gravitating body such as the Earth, density is highest at the core and decreases toward the surface: every mass element attracts every other, so the equilibrium is centrally condensed. Why is the nucleus different?

Key idea

Nuclear forces are *short-ranged*. A given nucleon interacts significantly only with its nearest neighbours, not with all other nucleons in the system. There is therefore no long-range "pull toward the centre," and the equilibrium density is set locally by the balance of short-range attraction and short-range repulsion (the hard core / Pauli pressure). The result is a saturated, roughly constant bulk density.

Remark

The range of the strong force is of order the pion Compton wavelength, $\hbar/(m_\pi c) \approx 1.4\,\text{fm}$—comparable to the mean inter-nucleon spacing at saturation. Beyond two or three fermis the force has essentially died away. That is why adding nucleons increases the *volume* at fixed density rather than packing the centre more tightly, and why the surface thickness is a nearly universal few fermis set by the force range itself.

Historical note: Hideki Yukawa (1907–1981)

Hideki Yukawa, working in Osaka in 1934–1935, proposed that the short-range nuclear force is mediated by the exchange of a massive boson. The resulting potential,

$$V(r) \propto \frac{e^{-r/\lambda}}{r}, \qquad \lambda = \frac{\hbar}{mc},$$

falls exponentially with a range fixed by the mediator mass m. Yukawa estimated $m \sim 100\,\mathrm{MeV}/c^2$; the pion, discovered in 1947, has $m_\pi \approx 140\,\mathrm{MeV}/c^2$ and $\lambda \approx 1.4\,\mathrm{fm}$. He received the 1949 Nobel Prize in Physics, the first Nobel awarded to a Japanese scientist. Yukawa's insight is the microscopic reason the nuclear density saturates: the force simply does not reach across the whole nucleus.

1.4 A second method: nuclear size from atomic energy shifts

Electron scattering is not the only route to a nuclear radius. Atomic electrons themselves sample the nuclear interior: their wave functions are nonzero at $r = 0$, so a finite nuclear size shifts atomic energy levels relative to the point-nucleus idealisation. Measuring those shifts, in ordinary or more sensitively muonic atoms, provides an independent determination of $\langle r^2 \rangle_{\mathrm{ch}}$.

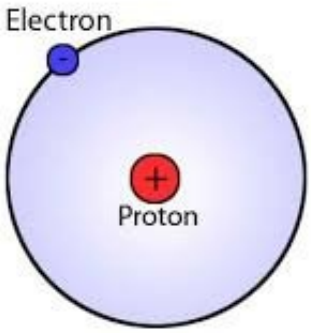

Figure 1.4: Schematic hydrogen-like atom: a single electron bound to a point-like positive charge. When the nucleus has finite size, the potential inside $r < R$ is modified and the bound-state energies shift.

1.4.1 Point nucleus: the Coulomb problem

For a hydrogen-like ion (one electron, nuclear charge Ze), the potential energy of the electron in the field of a *point* nucleus is

$$U(r) = -\frac{Ze^2}{4\pi\varepsilon_0\, r}. \tag{1.14}$$

The Hamiltonian is

$$H_0 = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{Ze^2}{4\pi\varepsilon_0\, r}, \tag{1.15}$$

and the ground-state energy of hydrogen ($Z = 1$) is the familiar

$$E_1 = -13.6\,\mathrm{eV}. \tag{1.16}$$

The corresponding ground-state wave function,

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}, \tag{1.17}$$

with Bohr radius $a_0 = 4\pi\varepsilon_0\hbar^2/(m_e e^2) \approx 0.529\,\text{Å}$, is nonzero at the origin:

$$|\psi_{100}(0)|^2 = \frac{Z^3}{\pi a_0^3} \neq 0. \tag{1.18}$$

The electron therefore has a finite probability of being found inside the nuclear volume. If the nucleus is extended rather than point-like, the potential inside $r < R$ differs from Equation (1.14), and the energy levels shift.

Caution

Some lecture notes write $\psi_{100} \propto Z^3/(\pi a_0^3)\, e^{-Zr/a_0}$ without the square root. The correctly normalised ground state is Equation (1.17); what is proportional to $Z^3/(\pi a_0^3)$ is $|\psi(0)|^2$, not ψ itself.

Historical note: Niels Bohr (1885–1962)

Niels Bohr introduced the first quantum model of the hydrogen atom in 1913, postulating discrete orbits with angular momentum $n\hbar$ and explaining the Rydberg formula for spectral lines. The Bohr radius a_0 that appears in Equation (1.17) is the radius of the $n = 1$ orbit in that model; in modern quantum mechanics it sets the exponential scale of the ground-state wave function. Bohr received the 1922 Nobel Prize in Physics for his services to the investigation of the structure of atoms and of the radiation emanating from them. The finite-size corrections we derive below are small perturbations on the Bohr–Schrödinger spectrum, but they are measurable, and they encode the nuclear radius.

1.4.2 Finite nucleus: electrostatics of a uniformly charged sphere

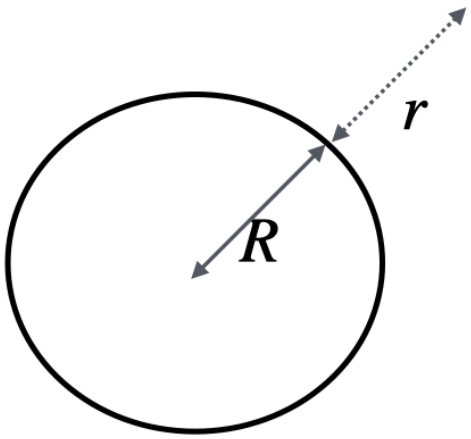

Figure 1.5: Spherical nucleus of radius R. Outside ($r > R$) the potential is identical to that of a point charge Ze. Inside ($r \leq R$) the field and potential must be recomputed for a uniform charge distribution.

Assume a spherical nucleus of radius R with charge Ze distributed uniformly. Gauss's law

gives the electric field.

Outside ($r > R$). All charge is enclosed, so

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{r^2}, \qquad V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{r} \quad (r > R), \tag{1.19}$$

identical to the point-charge result (taking $V(\infty) = 0$).

Inside ($r \le R$). The charge enclosed within radius r is $Ze \cdot (r^3/R^3)$, so

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Ze\,r}{R^3}, \qquad r \le R. \tag{1.20}$$

The potential is continuous at $r = R$. Integrating inward from the surface,

$$\begin{aligned} V(r) &= V(R) - \int_R^r E(r')\,\mathrm{d}r' = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{R} + \int_r^R \frac{1}{4\pi\varepsilon_0}\frac{Ze\,r'}{R^3}\,\mathrm{d}r' \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Ze}{R} + \frac{1}{4\pi\varepsilon_0}\frac{Ze}{R^3}\cdot\frac{1}{2}(R^2 - r^2) \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Ze}{2R^3}(3R^2 - r^2) = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{2R}\left(3 - \frac{r^2}{R^2}\right). \end{aligned} \tag{1.21}$$

The potential energy of an electron (charge $-e$) inside the nucleus is therefore

$$U(r) = -e\,V(r) = -\frac{Ze^2}{8\pi\varepsilon_0\,R}\left(3 - \frac{r^2}{R^2}\right), \qquad r \le R. \tag{1.22}$$

Definition: electron potential energy for a uniform sphere

$$U(r) = \begin{cases} -\dfrac{Ze^2}{8\pi\varepsilon_0\,R}\left(3 - \dfrac{r^2}{R^2}\right), & r \le R, \\[2ex] -\dfrac{Ze^2}{4\pi\varepsilon_0\,r}, & r > R. \end{cases}$$

Outside, nothing changes. Inside, the potential is finite at $r = 0$ and shallower than the pure Coulomb singularity.

1.4.3 Perturbation theory for the finite-size shift

Write the full Hamiltonian as $H = H_0 + H_1$, where H_0 is the point-nucleus operator Equation (1.15) and H_1 is the difference between the finite-size and point-nucleus potentials. Since the two potentials agree for $r > R$,

$$H_1(r) = \begin{cases} U_{\text{finite}}(r) - U_{\text{point}}(r), & r \le R, \\ 0, & r > R. \end{cases} \tag{1.23}$$

Explicitly, for $r \le R$,

$$H_1(r) = -\frac{Ze^2}{8\pi\varepsilon_0\, R}\left(3 - \frac{r^2}{R^2}\right) + \frac{Ze^2}{4\pi\varepsilon_0\, r} = \frac{Ze^2}{8\pi\varepsilon_0}\left[\frac{2}{r} - \frac{1}{R}\left(3 - \frac{r^2}{R^2}\right)\right]. \tag{1.24}$$

Caution

A common slip is to write the first term of H_1 as $Ze^2/(8\pi\varepsilon_0 r)$ rather than $Ze^2/(4\pi\varepsilon_0 r)$. The point-nucleus potential energy is $-Ze^2/(4\pi\varepsilon_0 r)$; when one forms $U_{\text{finite}} - U_{\text{point}}$, the point piece contributes $+Ze^2/(4\pi\varepsilon_0 r) = +2\,Ze^2/(8\pi\varepsilon_0 r)$. Using the wrong factor of two spoils the cancellation that makes $H_1(R) = 0$ (continuity of the potential at the nuclear surface).

In first-order perturbation theory the energy shift of an unperturbed state $|\psi_0\rangle$ is

$$\Delta E = \langle \psi_0 | H_1 | \psi_0 \rangle = \int \psi_0^*(\boldsymbol{r})\, H_1(r)\, \psi_0(\boldsymbol{r})\, \mathrm{d}^3 r. \tag{1.25}$$

Because H_1 is supported only inside the tiny nuclear volume, and because s-state wave functions vary slowly on the scale of R, one may approximate $\psi_0(\boldsymbol{r}) \approx \psi_0(0)$ inside the integral. Then

$$\Delta E \approx |\psi_0(0)|^2 \int_{r<R} H_1(r)\, \mathrm{d}^3 r. \tag{1.26}$$

The radial integral can be evaluated in closed form. With $k \equiv Ze^2/(4\pi\varepsilon_0)$ one has $H_1(r) = k\big[1/r - 3/(2R) + r^2/(2R^3)\big]$ for $r \le R$, so

$$\int_{r<R} H_1\, \mathrm{d}^3 r = 4\pi k \int_0^R \left(r - \frac{3r^2}{2R} + \frac{r^4}{2R^3}\right)\mathrm{d}r = 4\pi k\left[\frac{R^2}{2} - \frac{R^2}{2} + \frac{R^2}{10}\right] = \frac{2\pi}{5}\, k\, R^2. \tag{1.27}$$

(The first two terms cancel; only the r^4 piece survives.) Hence

$$\Delta E \approx \frac{2\pi}{5}\frac{Ze^2}{4\pi\varepsilon_0} R^2 |\psi_0(0)|^2 = \frac{2\pi}{5}\frac{Ze^2}{4\pi\varepsilon_0} R^2 \frac{Z^3}{\pi a_0^3} \tag{1.28}$$

for the ground state (using Equation (1.18)).

More generally, for an arbitrary spherical charge distribution the leading finite-size shift depends only on the mean-square charge radius. To see why, expand the interior potential difference for r small compared with the atomic scale. Outside a spherical distribution the potential is pure Coulomb; inside, Gauss's law gives

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q(r)}{r^2}, \qquad q(r) = \int_0^r 4\pi r'^2 \rho_{\text{ch}}(r')\, \mathrm{d}r'.$$

Integrating from infinity and subtracting the point-nucleus potential yields, after one partial

integration,

$$\Delta E = \frac{2\pi}{3}\frac{Ze^2}{4\pi\varepsilon_0}\langle r^2\rangle_{\rm ch}|\psi(0)|^2 + \cdots , \tag{1.29}$$

with $\langle r^2\rangle_{\rm ch} = (1/Ze)\int \rho_{\rm ch}(r)\, r^2\, \mathrm{d}^3r$. For a uniform sphere $\langle r^2\rangle = \frac{3}{5}R^2$, and Equation (1.29) reduces exactly to Equation (1.28):

$$\frac{2\pi}{3}k\cdot\frac{3}{5}R^2|\psi(0)|^2 = \frac{2\pi}{5}kR^2|\psi(0)|^2.$$

This is the form used in precision atomic physics and muonic-atom analyses.

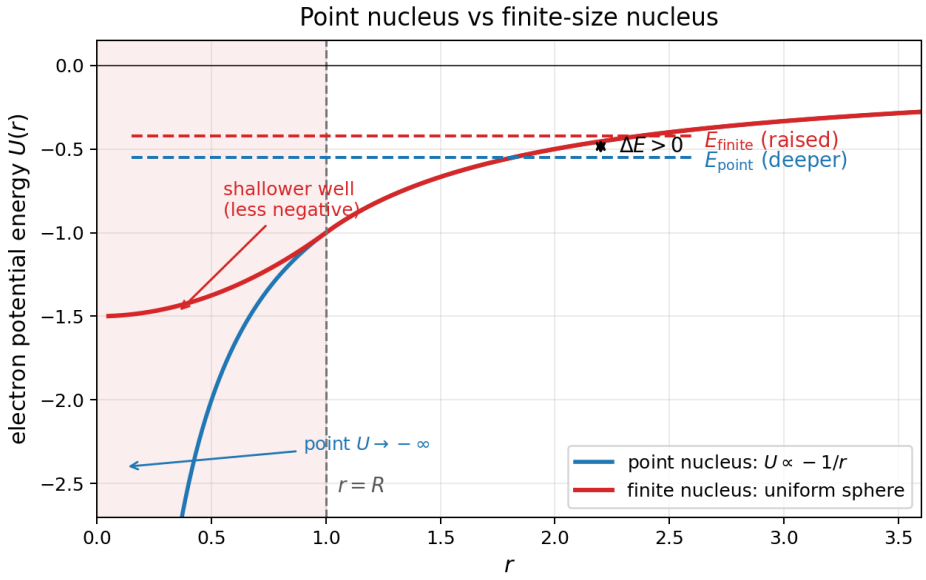

Figure 1.6: Point Coulomb well ($U \propto -1/r$, blue) versus a uniform sphere of radius R (red). Inside the nucleus the finite well is shallower. Dashed lines: schematic bound-state ($1s$) energies; the finite-size level lies higher by $\Delta E > 0$. Outside $r = R$ the potentials agree.

Remark

Because $H_1 > 0$ inside the nucleus (the finite-size well is shallower than pure Coulomb; Figure 1.6), one has $\Delta E > 0$: the **bound-state energy** E_{1s} moves *upward* (binding decreases). Only s waves have $|\psi(0)| \neq 0$ in the non-relativistic theory, so $p, d, \ldots$ states feel a much smaller finite-size effect at this order.

Key idea

A finite nuclear size *raises* the bound-state energy of s states (E_{1s} less negative; Figure 1.6), because the electron, when inside the nucleus, feels a shallower potential than the pure Coulomb well. The shift scales as Z^4R^2/a_0^3 for hydrogen-like ions and is largest for heavy nuclei and for muons (whose Bohr radius is ~ 200 times smaller).

Example: order of magnitude for hydrogen

For hydrogen, $Z = 1$, $R \sim 1\,\text{fm} = 10^{-5}\,a_0$, and $|\psi(0)|^2 = 1/(\pi a_0^3)$. Then

$$\frac{\Delta E}{|E_1|} \sim \left(\frac{R}{a_0}\right)^2 \sim 10^{-10}.$$

The shift is tiny in ordinary hydrogen (parts in 10^{10}) but becomes measurable in high-Z electronic atoms and is dramatic in muonic atoms, where the muon wave function overlaps the nucleus strongly [2].

Historical note: muonic atoms and the proton radius

Replacing the electron by a muon multiplies the reduced mass by $m_\mu/m_e \approx 207$, shrinks all Bohr radii by the same factor, and amplifies finite-size shifts by $\sim (207)^3$ in the wave-function density at the origin. Precision spectroscopy of muonic hydrogen at the Paul Scherrer Institute yielded a proton charge radius $r_p \approx 0.84\,\text{fm}$, about 4% smaller than the value long accepted from electronic hydrogen and early electron scattering: the so-called *proton radius puzzle*. The episode underlines both the power of atomic probes of nuclear size and the care required in comparing different experimental methods [2].

Historical note: Erwin Schrödinger (1887–1961)

Erwin Schrödinger formulated wave mechanics in 1926, writing the eigenvalue equation $H\psi = E\psi$ that underlies Equations (1.15) and (1.17). His treatment of the hydrogen atom recovered the Bohr levels and supplied the wave functions needed for perturbation theory. He shared the 1933 Nobel Prize in Physics with Paul Dirac. The finite-size analysis of this section is ordinary first-order perturbation theory on Schrödinger's hydrogen Hamiltonian, a direct descendant of that 1926 paper.

The finite-size shift derived here is the second experimental route to nuclear size: the same $\langle r^2\rangle_{\text{ch}}$ that Lecture 1 extracted from the low-q slope of $F(q)$ now appears as an atomic energy correction. Absolute shifts are hard to isolate because no point-nucleus reference sample exists; Lecture 3 shows how isotope differences cancel everything except ΔR.

1.5 Summary and outlook

- Electron scattering measures $\rho_p(r)$. Under $\rho_n/\rho_p = N/Z$, the nucleon density is $\rho = (A/Z)\,\rho_p$.
- Rescaled densities for different nuclei share a common central value $\rho_0 \approx 0.16/\text{fm}^3$: nuclear matter saturates.
- The Woods–Saxon surface thickness is $t = 4a\ln 3$ with the usual $a = 0.52\,\text{fm} \pm 0.01\,\text{fm}$ ($t \approx 2.3\,\text{fm}$); the half-density radius obeys $R_0 = 1.2\,\text{fm}\,A^{1/3}$ [1].
- A flat bulk density is the geometric signature of a short-range strong force: nucleons feel only their neighbours.
- Atomic electrons (and especially muons) provide a second radius measurement via the finite-size shift $\Delta E \propto Z\,|\psi(0)|^2\,\langle r^2\rangle_{\text{ch}}$.

Next. Lecture 3 specialises the finite-size shift to isotope differences and muonic atoms,

recovering the same $R_0 \approx 1.2\,\mathrm{fm}$ from spectroscopy. Lectures 4–5 then leave size for binding energy and the SEMF that encodes saturation as a volume term.

1.6 Deeper physics: saturation density and short-range force

1.6.1 Numerical saturation density

If the mass density is roughly uniform inside $R = r_0 A^{1/3}$ with $r_0 \approx 1.2\,\mathrm{fm}$, the mean nucleon density is

$$\rho_0 = \frac{A}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi r_0^3} \approx 0.14\text{–}0.17\,\mathrm{fm}^{-3}, \tag{1.30}$$

depending on the precise radius definition (half-density vs equivalent uniform sphere vs rms). A representative value is $\rho_0 \approx 0.16\,\mathrm{fm}^{-3}$ [1, 2, 4]. In mass units this is

$$\bar{\rho} \sim 2.3 \times 10^{17}\,\mathrm{kg\,m^{-3}} \sim 2 \times 10^{14}\,\mathrm{g\,cm^{-3}}, \tag{1.31}$$

about 10^{14} times liquid water: the densest stable bulk matter on Earth is ordinary nuclear matter.

1.6.2 Why density saturation implies a short-range force

If every nucleon interacted attractively with every other nucleon (infinite-range two-body force), the binding energy would grow as $\binom{A}{2} \propto A^2$ and the equilibrium density would not be A-independent. Observed $B \propto A$ (Lecture 4) and $\rho \approx$ const together require that each nucleon feels only a fixed number of neighbours: the force range is of order the internucleon spacing $\sim$ 1–2 fm.

Light vs heavy surface fraction

With $t \approx 2.3\,\mathrm{fm}$ fixed and $R = 1.2\,A^{1/3}\,\mathrm{fm}$, a light nucleus ($R \sim 3\,\mathrm{fm}$) is mostly surface, while lead ($R \sim 7\,\mathrm{fm}$) is mostly bulk. That geometric fact foreshadows the SEMF: surface energy dominates the rise of B/A at low A; volume energy dominates the mid-mass plateau [1].

Remark

Mass vs charge distributions. Matter densities inferred from hadronic probes are similar in shape to charge densities, but not identical: neutron skins in heavy nuclei make the matter radius slightly larger than the charge radius. Lecture 1's electron probe measures charge; Lecture 2's saturation argument concerns the *matter* density that sets the SEMF volume term.

1.7 Further physics: saturation and the uncertainty principle

Constant central density and $B/A \sim$ const are two faces of the same fact: each nucleon interacts mainly with nearest neighbours (**saturation** of the nuclear force). A quantum link makes that

plausible. If A nucleons share a volume $\propto R^3$ with $R = r_0 A^{1/3}$, each occupies a linear size $\Delta x \sim r_0$. The uncertainty principle then gives a mean kinetic energy per nucleon of order

$$\langle T \rangle \sim \frac{\hbar^2}{2m_N r_0^2} \sim \text{tens of MeV}, \tag{1.32}$$

nearly independent of A. Because the kinetic energy per particle does not grow with system size, neither does the net binding per nucleon once nearest-neighbour attraction is fixed [14].

That argument fails for atoms: the long-range Coulomb field of the nucleus pulls electrons tighter as Z grows, so atomic size does not scale as $Z^{1/3}$ and the electron density rises with Z. Nuclear forces do not act that way: density saturates.

1.7.1 Saturation requires more than the Pauli principle

Nearest-neighbour counting explains why density and B/A do not grow with A, but a complete picture of saturation needs three ingredients [14]:

1. short-range attraction ($r \sim 1\,\text{fm}$);
2. a repulsive "hard core" at $r \lesssim 0.5\,\text{fm}$ (partly Pauli/quark structure);
3. the Pauli principle for nucleons as fermions.

Any pure power-law attraction would not saturate. The empirical Woods–Saxon density (flat interior, thin surface) is the geometric fingerprint of that three-part force.

1.7.2 Atoms vs nuclei

Atomic size does *not* grow as $Z^{1/3}$: the long-range Coulomb field of the nucleus increases electron density with Z. Nuclear density is approximately constant because the force is short-ranged and saturates. That contrast is the essential reason liquid-drop thinking works for nuclei and not for atoms [14, 5].

1.7.3 Fermi momentum and density at saturation

Treat protons and neutrons as independent Fermi gases in a volume $V = (4\pi/3)R^3$ with bulk density

$$n_0 \approx 0.15\text{–}0.17\,\text{fm}^{-3} \tag{1.33}$$

[14, 1]. For $N \approx Z \approx A/2$ the Fermi energy of each species is

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \approx 35\,\text{MeV}, \tag{1.34}$$

corresponding to

$$p_F \approx 265\,\text{MeV}/c, \qquad k_F = p_F/\hbar \approx 1.33\,\text{fm}^{-1}. \tag{1.35}$$

The mean kinetic energy per nucleon in a free Fermi gas is $\frac{3}{5}\varepsilon_F \approx 21\,\text{MeV}$. Neutron-scattering and optical-model analyses place the depth of the mean field near $U \sim -40\,\text{MeV}$; the net volume

binding is then of order $a_V \sim |U| - \frac{3}{5}\varepsilon_F \sim 15$–$20\,\mathrm{MeV}$, consistent with the SEMF volume coefficient [14].

These numbers fix the density and energy scales that Lectures 4–6 carry into the liquid-drop formula and into compact-star estimates.

1.8 Problems with solutions

Problem 2.1 — Saturation density

Using $R = 1.2\,A^{1/3}\,\mathrm{fm}$, compute the mean nucleon number density $\rho_0 = A/(\frac{4}{3}\pi R^3)$ in fm^{-3} and in $\mathrm{g\,cm}^{-3}$ (take $m_N = 1.67 \times 10^{-24}\,\mathrm{g}$).

Solution 2.1

$$\rho_0 = \frac{3}{4\pi r_0^3} = \frac{3}{4\pi(1.2)^3} \approx 0.138\,\mathrm{fm}^{-3} \sim 0.14\text{–}0.17\,\mathrm{fm}^{-3}$$

depending on the precise radius definition. In mass units,

$$\rho_0 m_N \approx 2.3 \times 10^{14}\,\mathrm{g\,cm}^{-3},$$

as in Problem 1.1. This is the saturation density underlying the SEMF volume term.

Problem 2.2 — Neutron-star radius from density (Petrera 1.1.3)

A neutron star has mass $M \sim M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ and density comparable to nuclear density $\rho \sim 2.3 \times 10^{14}\,\mathrm{g\,cm}^{-3}$. Estimate its radius.

Solution 2.2

$$\frac{4}{3}\pi R^3 \rho = M \quad \Rightarrow \quad R = \left(\frac{3M}{4\pi\rho}\right)^{1/3} \approx \left(\frac{3 \times 2 \times 10^{33}\,\mathrm{g}}{4\pi \times 2.3 \times 10^{14}\,\mathrm{g\,cm}^{-3}}\right)^{1/3} \approx 13\,\mathrm{km}.$$

Nuclear saturation density plus solar mass already yields the observed $R \sim 10\,\mathrm{km}$ scale (Lecture 7).

Problem 2.3 — Surface thickness

For a Woods–Saxon profile with $a = 0.52\,\mathrm{fm}$, compute the skin thickness t over which the density falls from 90% to 10% of its central value.

Solution 2.3

$$t = 4a \ln 3 \approx 4 \times 0.52 \times 1.099 \approx 2.3\,\mathrm{fm}.$$

The surface thickness is nearly independent of A, while $R \propto A^{1/3}$ grows: light nuclei are almost all surface; heavy nuclei have a thin skin on a large bulk.

Bibliography

[1] R. A. Dunlap, *An Introduction to the Physics of Nuclei and Particles* (2nd ed.), IOP Publishing, Bristol (2023), DOI: 10.1088/978-0-7503-6094-4, Ch. 3 (Nuclear composition and size).

[2] W. Demtröder, *Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2023).

[3] S. Petrera, *Problems and Solutions in Nuclear and Particle Physics*, UNITEXT for Physics, Springer, Cham (2019).

[4] F. Terranova, *A Modern Primer in Particle and Nuclear Physics*, Oxford University Press (2021).

[5] S. D'Auria, *Introduction to Nuclear and Particle Physics*, Undergraduate Lecture Notes in Physics, Springer, Cham (2018).

[6] K. S. Krane, *Introductory Nuclear Physics*, Wiley, New York (1988).

[7] R. Hofstadter, "Electron scattering and nuclear structure," Rev. Mod. Phys. **28**, 214 (1956).

[8] H. F. Ehrenberg, R. Hofstadter, U. Meyer-Berkhout, D. G. Ravenhall, and S. E. Sobottka, "High-Energy Electron Scattering and the Charge Distribution of Carbon-12 and Oxygen-16," Phys. Rev. **113**, 666 (1959).

[9] H. Yukawa, "On the interaction of elementary particles. I," Proc. Phys.-Math. Soc. Japan **17**, 48 (1935).

[10] R. D. Woods and D. S. Saxon, "Diffuse surface optical model for nucleon-nuclei scattering," Phys. Rev. **95**, 577 (1954).

[11] H. G. J. Moseley, "The high-frequency spectra of the elements," Phil. Mag. **26**, 1024 (1913); **27**, 703 (1914).

[12] C. F. von Weizsäcker, "Zur Theorie der Kernmassen," Z. Phys. **96**, 431 (1935).

[13] P. Cappellaro, *Introduction to Applied Nuclear Physics* (MIT OpenCourseWare / LibreTexts), CC BY-NC-SA 4.0.

[14] J.-L. Basdevant, J. Rich, and M. Spiro, *Fundamentals in Nuclear Physics: From Nuclear Structure to Cosmology*, Springer, New York (2005).

[15] D. Tong, *Particle Physics*, Lecture notes (University of Cambridge / CERN school notes).