Chapter 1

Most Stable Z and Mass Parabolas

"For constant A, the mass formula represents a parabola of M vs Z."

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1.1 Atomic mass from the SEMF

Lecture 4 constructed the binding energy $B(A, Z)$ and explained how its five terms shape the B/A curve. Stability against β decay is decided not by B alone but by the *atomic* mass $M(A, Z)$: at fixed mass number A, the isobar with the lowest mass cannot lower its energy by converting a neutron into a proton (or vice versa). Writing M as a function of Z at fixed A produces the mass parabolas of this lecture [6, 1, 14].

Key idea

From binding energy to atomic mass.

$$M(A, Z)\, c^2 = [Z\, m(^1\mathrm{H}) + (A - Z)\, m_n]c^2 - B(A, Z).$$

β decay changes Z by ± 1 at fixed A, so it moves along an isobaric slice of this surface. The SEMF turns that slice into a parabola in Z.

The SEMF binding energy is

$$B(A, Z) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_\mathrm{a} \frac{(N-Z)^2}{A} + \delta(A, Z), \tag{1.1}$$

with $N = A - Z$ and

$$\delta = \begin{cases} +a_P A^{-3/4} & \text{even–even,} \\ 0 & \text{odd } A, \\ -a_P A^{-3/4} & \text{odd–odd.} \end{cases} \tag{1.2}$$

The atomic mass energy is

$$M(A, Z)\, c^2 = [Z\, m(^1\mathrm{H}) + (A - Z)\, m_n]c^2 - B(A, Z). \tag{1.3}$$

Using $m(^1\mathrm{H})$ rather than m_p folds the electron mass into the atomic ledger, which is what β

energetics require. Equivalently one may write the free-nucleon piece as $(A-Z)m_n+Z(m_p+m_e)$; the difference $m(^1\mathrm{H})-(m_p+m_e)$ is the electron binding in hydrogen and is negligible at the MeV scale used here [6, 1].

Key idea

Which SEMF terms matter at fixed A? When A is held constant, the volume term $a_V A$ and the surface term $a_S A^{2/3}$ are the same for every isobar and drop out of mass *differences*. What remains are:

- the free-nucleon mass piece, linear in Z through $Z\,m(^1\mathrm{H})+(A-Z)\,m_n$;
- Coulomb, quadratic in Z;
- asymmetry, quadratic in Z through $(A-2Z)^2$;
- pairing δ, constant on each even/odd Z ladder.

Lecture 4 fixed the bulk physics of B/A; Lecture 5 extracts the Z dependence that controls β stability.

Substitute (1.1) into (1.3). Because $M\,c^2$ contains $-B$, every binding term changes sign:

$$\begin{aligned} M(A,Z)\,c^2 =& [Z\,m(^1\mathrm{H})+(A-Z)\,m_n]c^2 \\ & -a_V A + a_S A^{2/3} + a_C\frac{Z(Z-1)}{A^{1/3}} + a_\mathrm{a}\frac{(A-2Z)^2}{A} - \delta. \end{aligned} \tag{1.4}$$

Here $N-Z=A-2Z$ has already been used in the asymmetry piece.

1.2 Why $M(A,Z)$ is a parabola in Z

Rewrite each Z-dependent contribution as a polynomial in Z. First the free-nucleon (atomic) piece:

$$[Z\,m(^1\mathrm{H})+(A-Z)\,m_n]c^2 = A\,m_n c^2 + [m(^1\mathrm{H})-m_n]c^2\,Z. \tag{1.5}$$

Next the asymmetry energy. Expand $(A-2Z)^2=A^2-4AZ+4Z^2$:

$$\begin{aligned} a_\mathrm{a}\frac{(A-2Z)^2}{A} &= \frac{a_\mathrm{a}}{A}(A^2-4AZ+4Z^2) \\ &= a_\mathrm{a}A - 4a_\mathrm{a}Z + \frac{4a_\mathrm{a}}{A}\,Z^2. \end{aligned} \tag{1.6}$$

For Coulomb, distribute $Z(Z-1)=Z^2-Z$:

$$a_C\frac{Z(Z-1)}{A^{1/3}} = \frac{a_C}{A^{1/3}}\,Z^2 - \frac{a_C}{A^{1/3}}\,Z. \tag{1.7}$$

Volume $-a_V A$ and surface $+a_S A^{2/3}$ depend only on A. Pairing δ is constant on a given even/odd ladder of Z and will be kept separate from the smooth polynomial.

Insert (1.5)–(1.7) into (1.4):

$$
\begin{aligned}
M(A,Z)\,c^2 = &\Big(A\,m_n c^2 - a_V A + a_S A^{2/3} + a_\mathrm{a} A\Big) \\
&+ \Big([m(^1\mathrm{H}) - m_n]c^2 - 4a_\mathrm{a} - \frac{a_C}{A^{1/3}}\Big) Z \\
&+ \Big(\frac{a_C}{A^{1/3}} + \frac{4a_\mathrm{a}}{A}\Big) Z^2 - \delta.
\end{aligned}
\tag{1.8}
$$

This is the quadratic form

$$
M(A,Z)\,c^2 = \alpha(A) + \beta(A)\,Z + \gamma(A)\,Z^2 - \delta,
\tag{1.9}
$$

with coefficients (at fixed A)

$$
\alpha(A) = A\,m_n c^2 - a_V A + a_S A^{2/3} + a_\mathrm{a} A,
\tag{1.10}
$$

$$
\beta(A) = [m(^1\mathrm{H}) - m_n]c^2 - 4a_\mathrm{a} - \frac{a_C}{A^{1/3}},
\tag{1.11}
$$

$$
\gamma(A) = \frac{a_C}{A^{1/3}} + \frac{4a_\mathrm{a}}{A}.
\tag{1.12}
$$

Both contributions to γ are positive for physical coefficients, so $\gamma(A) > 0$: the parabola opens upward. The constant $\alpha(A)$ shifts the whole curve uniformly and does not affect which Z is most stable. Nuclei far from the vertex have higher mass and can β decay toward the minimum [6, 1].

Differentiating (1.9) (pairing held fixed) gives

$$
\frac{\partial}{\partial Z}\big[M(A,Z)\,c^2\big] = \beta(A) + 2\gamma(A)\,Z.
\tag{1.13}
$$

Setting the derivative to zero yields the continuous location of the vertex,

$$
Z_\mathrm{min} = -\frac{\beta(A)}{2\gamma(A)}.
\tag{1.14}
$$

Substitute (1.11)–(1.12):

$$
\begin{aligned}
Z_\mathrm{min} &= \frac{-[m(^1\mathrm{H}) - m_n]c^2 + 4a_\mathrm{a} + a_C A^{-1/3}}{2\big(a_C A^{-1/3} + 4a_\mathrm{a}/A\big)} \\
&= \frac{(m_n - m(^1\mathrm{H}))c^2 + a_C A^{-1/3} + 4a_\mathrm{a}}{2a_C A^{-1/3} + 8a_\mathrm{a}/A},
\end{aligned}
\tag{1.15}
$$

which is the same expression obtained by differentiating (1.4) term by term in the next section.

Geometrically, Coulomb raises the mass of proton-rich isobars; asymmetry raises the mass when $|N - Z|$ is large. The minimum is the compromise charge where those penalties balance, moderated by the small linear term from nucleon mass differences and the linear piece of the Coulomb expansion.

The most stable isobar is the integer Z nearest the minimum of that parabola.

Key idea

At fixed A, Coulomb repulsion and asymmetry energy together make the mass a quadratic in Z. Minimising mass maximises stability against β decay.

1.3 Most stable Z: differentiation of the mass formula

Ignore pairing for the moment (exact for odd A; for even A the pairing shifts two parallel parabolas but does not move their common $Z_{\min}$ at leading order). Differentiate (1.4) with respect to Z at fixed A.

The free-nucleon term contributes

$$\frac{\partial}{\partial Z}\left[Z\, m(^1\mathrm{H}) + (A-Z)\, m_n\right]c^2 = \left[m(^1\mathrm{H}) - m_n\right]c^2. \tag{1.16}$$

For Coulomb,

$$\frac{\partial}{\partial Z}\left[a_C \frac{Z(Z-1)}{A^{1/3}}\right] = \frac{a_C}{A^{1/3}}\,(2Z-1). \tag{1.17}$$

For asymmetry, with $u = A - 2Z$ so $\partial u/\partial Z = -2$,

$$\frac{\partial}{\partial Z}\left[a_\mathrm{a} \frac{(A-2Z)^2}{A}\right] = \frac{a_\mathrm{a}}{A}\cdot 2(A-2Z)\cdot(-2) = -\frac{4a_\mathrm{a}}{A}(A-2Z). \tag{1.18}$$

Volume, surface, and (for now) pairing do not contribute. Adding these derivatives,

$$\frac{\partial}{\partial Z}\left[M(A,Z)\,c^2\right] = \left[m(^1\mathrm{H}) - m_n\right]c^2 + a_C\frac{(2Z-1)}{A^{1/3}} - a_\mathrm{a}\frac{4(A-2Z)}{A}. \tag{1.19}$$

Set the derivative to zero and rearrange. Writing $\left[m(^1\mathrm{H}) - m_n\right]c^2 = -\left[m_n - m(^1\mathrm{H})\right]c^2$,

$$a_C\frac{(2Z-1)}{A^{1/3}} - \left[m_n - m(^1\mathrm{H})\right]c^2 = 4a_\mathrm{a}\frac{(A-2Z)}{A}. \tag{1.20}$$

Expand the left- and right-hand sides of (1.20):

$$a_C\frac{2Z}{A^{1/3}} - \frac{a_C}{A^{1/3}} - \left[m_n - m(^1\mathrm{H})\right]c^2 = 4a_\mathrm{a} - \frac{8a_\mathrm{a}}{A}\,Z, \tag{1.21}$$

where the asymmetry identity

$$\frac{4a_\mathrm{a}}{A}(A-2Z) = 4a_\mathrm{a} - \frac{8a_\mathrm{a}}{A}\,Z$$

has been used. Move every term that multiplies Z to the left and every constant to the right:

$$\frac{2a_C}{A^{1/3}}\,Z + \frac{8a_\mathrm{a}}{A}\,Z = \left[m_n - m(^1\mathrm{H})\right]c^2 + \frac{a_C}{A^{1/3}} + 4a_\mathrm{a}, \tag{1.22}$$

and therefore

$$Z_{\min} = \frac{(m_n - m(^1\mathrm{H}))c^2 + a_C A^{-1/3} + 4a_\mathrm{a}}{2a_C A^{-1/3} + 8a_\mathrm{a}/A}. \tag{1.23}$$

This matches (1.15) and is the exact stationary condition for the SEMF mass parabola [6, 1]. The integer Z closest to $Z_{\min}$ is the smooth-SEMF candidate; pairing, shell structure, and evaluated atomic masses decide the actual beta-stable member.

1.3.1 Numerical size of the free-mass term

From atomic mass units,

$$m_n - m(^1\mathrm{H}) = 1.008665 - 1.007825 = 0.000840\,\mathrm{u}, \tag{1.24}$$

so

$$\delta_{nH} \equiv \left[m_n - m(^1\mathrm{H})\right]c^2 \approx 0.000840 \times 931.5 \approx 0.782\,\mathrm{MeV}. \tag{1.25}$$

With the usual coefficients $a_C = 0.72\,\mathrm{MeV}$ and $a_\mathrm{a} = 23\,\mathrm{MeV}$, for $A \sim 100$ one has

$$\frac{a_C}{A^{1/3}} \approx 0.16\,\mathrm{MeV}, \qquad 4a_\mathrm{a} = 92\,\mathrm{MeV}, \qquad \frac{4a_\mathrm{a}}{A} \approx 0.92\,\mathrm{MeV}.$$

Thus δ_{nH} is not negligible next to $a_C/A^{1/3}$, but both are small compared with the asymmetry scale $4a_\mathrm{a}$ that dominates the numerator of (1.23) [6, 14].

1.3.2 Working formula used in the lecture

In the numerator of (1.23), drop δ_{nH} and $a_C A^{-1/3}$ relative to $4a_\mathrm{a}$. The relative error is of order $(0.782 + 0.16)/92 \sim 1\%$ at $A \sim 100$. Then

$$Z_\mathrm{min} \approx \frac{4a_\mathrm{a}}{2a_C A^{-1/3} + 8a_\mathrm{a}/A}. \tag{1.26}$$

Divide numerator and denominator by $8a_\mathrm{a}/A$. The pieces are

$$\frac{4a_\mathrm{a}}{8a_\mathrm{a}/A} = \frac{4a_\mathrm{a} \cdot A}{8a_\mathrm{a}} = \frac{A}{2}, \tag{1.27}$$

$$\frac{2a_C A^{-1/3}}{8a_\mathrm{a}/A} = \frac{2a_C A^{-1/3} \cdot A}{8a_\mathrm{a}} = \frac{a_C}{4a_\mathrm{a}}\,A^{2/3}, \tag{1.28}$$

$$\frac{8a_\mathrm{a}/A}{8a_\mathrm{a}/A} = 1, \tag{1.29}$$

so

$$Z_\mathrm{min} \approx \frac{A/2}{1 + \frac{a_C}{4a_\mathrm{a}}\,A^{2/3}} = \frac{A}{2\left(1 + \frac{a_C}{4a_\mathrm{a}}\,A^{2/3}\right)}. \tag{1.30}$$

With $a_C/a_\mathrm{a} = 0.72/23 \approx 0.0313$,

$$\frac{a_C}{4a_\mathrm{a}} = \frac{0.72}{4 \cdot 23} = \frac{0.72}{92} \approx 0.0078,$$

and therefore

$$Z_\mathrm{min} \approx \frac{A}{2\left(1 + 0.0078\,A^{2/3}\right)}. \tag{1.31}$$

The same approximated expression can be written without the factor of two in the denominator structure:

$$Z_\mathrm{min} \approx \frac{A}{2 + \left(a_C/(2a_\mathrm{a})\right)A^{2/3}}, \tag{1.32}$$

because

$$2\Big(1 + \tfrac{a_C}{4a_\mathrm{a}}A^{2/3}\Big) = 2 + \tfrac{a_C}{2a_\mathrm{a}}A^{2/3}, \tag{1.33}$$

and with the same coefficients $a_C/(2a_\mathrm{a}) = 0.72/46 \approx 0.0157$. A closely related empirical fit uses 0.0075 instead of 0.0078 in (1.31) [14].

Key idea

Light vs heavy nuclei. For small A, $0.0078\,A^{2/3} \ll 1$, so $Z_\mathrm{min} \approx A/2$ ($N \approx Z$): oxygen, carbon, calcium. For large A the Coulomb correction grows, $Z_\mathrm{min} < A/2$, and long-lived nuclei become neutron rich ($N/Z \sim 1.5$ for lead). Equation (1.31) gives $Z/A \approx 0.41$ for heavy species, matching observation [6, 1]. Without Coulomb, (1.23) reduces to $Z_\mathrm{min} = A/2$ up to the tiny δ_{nH} shift: every extra neutron in a heavy stable nucleus is a Coulomb effect competing with asymmetry.

A	Z_min (approx.)	Nearest stable Z	N/Z
16	7.6	8 (^{16}O)	1.0
40	18.3	18 (^{40}Ar); ^{40}Ca also stable	1.22 at $Z = 18$
56	25.1	26 (^{56}Fe)	1.2
125	52.3	52 (^{125}Te)	1.4
208	81.6	82 (^{208}Pb)	1.5

Values of Z_min come from (1.31). At $A = 40$ the smooth formula places the minimum near argon, not calcium. ^{40}Ca is nevertheless stable because its doubly magic shell closure supplies binding absent from the SEMF. The nearest integer to a smooth minimum is therefore a guide, not an evaluated-mass verdict.

1.4 Odd A: one parabola and β decay toward the minimum

For odd A, $\delta = 0$. There is a *single* mass parabola. Nuclei with Z far from Z_min have large mass differences to their neighbours and decay by $\beta^\pm$ or electron capture; nuclei near the minimum are stable or long-lived.

1.4.1 β energetics on the parabola

Along an odd-A isobar the available energy for a single β step is the atomic mass difference of neighbouring isobars. For β^- ($Z \to Z+1$),

$$Q_{\beta^-} = \big[M(A,Z) - M(A,Z+1)\big]c^2. \tag{1.34}$$

For β^+ ($Z \to Z-1$), the nuclear process creates a positron, so two electron masses must be accounted for when atomic masses are used:

$$Q_{\beta^+} = \big[M(A,Z) - M(A,Z-1)\big]c^2 - 2m_ec^2. \tag{1.35}$$

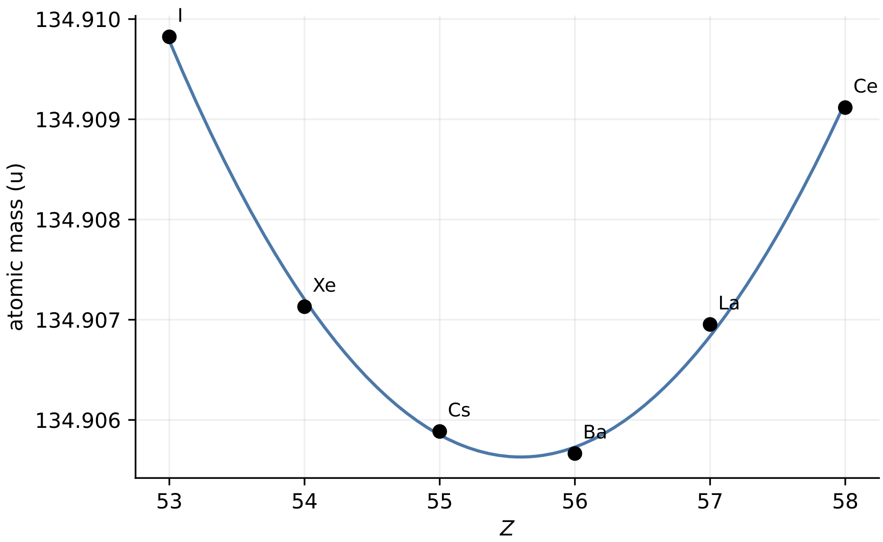

Figure 1.1: Mass parabola for odd $A = 135$ (data-generated from evaluated atomic masses; Dunlap tables transcribed for this course). One minimum near $Z = 56$; one (or few) β-stable isobar(s). Decay energy increases as one moves away from the minimum [1, 6].

Electron capture has a slightly lower threshold than β^+ (approximately (1.35) without the $2m_ec^2$ subtraction, apart from atomic-electron binding).

Apply the odd-A convention $Mc^2 = \alpha + \beta Z + \gamma Z^2$. First expand the neighbour:

$$\begin{aligned} M(A, Z+1)\, c^2 &= \alpha + \beta(Z+1) + \gamma(Z+1)^2 \\ &= \alpha + \beta Z + \beta + \gamma(Z^2 + 2Z + 1) \\ &= \alpha + \beta Z + \gamma Z^2 + \beta + 2\gamma Z + \gamma. \end{aligned} \tag{1.36}$$

Subtract from $M(A, Z)\, c^2$:

$$M(A, Z)\, c^2 - M(A, Z+1)\, c^2 = -\beta - 2\gamma Z - \gamma = -\beta - \gamma(2Z + 1). \tag{1.37}$$

From (1.14), $Z_{\min} = -\beta/(2\gamma)$, so $\beta = -2\gamma Z_{\min}$ and $-\beta = 2\gamma Z_{\min}$. Therefore

$$Q_{\beta^-} = 2\gamma Z_{\min} - \gamma(2Z + 1) = 2\gamma(Z_{\min} - Z - \tfrac{1}{2}). \tag{1.38}$$

Hence $Q_{\beta^-} > 0$ whenever $Z \le Z_{\min} - 1$ (integer steps below the vertex). Because $\gamma > 0$, the magnitude grows linearly as one climbs the sides of the parabola: β Q values increase away from $Z_{\min}$ [6, 1].

Key idea

Direction of β decay from geometry alone.

$Z < Z_{\min}$	neutron rich	β^- $(n \to p + e^- + \bar{\nu}_e)$,
$Z > Z_{\min}$	proton rich	β^+ or electron capture,
$Z \approx Z_{\min}$	stable / long-lived	no allowed single β.

The SEMF fixes the *direction* and approximate Q; Fermi theory fixes spectra and lifetimes.

1.5 Even A: pairing splits the mass surface into two parabolas

For even A, the pairing term shifts even–even nuclei *down* and odd–odd nuclei *up* by $|\delta| = a_P A^{-3/4}$ in the binding-energy convention of (1.1). Because $M\,c^2$ contains $-\delta$, the mass of an even–even isobar is lowered by $|\delta|$ and that of an odd–odd isobar is raised by $|\delta|$. The vertical separation between the two parabolas is therefore $2|\delta|$. The mass surface becomes two parallel parabolas sharing the same $Z_{\min}$ but offset in energy [6, 1, 14]:

$$M_{\rm ee}\,c^2 = \alpha + \beta Z + \gamma Z^2 - |\delta|, \tag{1.39}$$

$$M_{\rm oo}\,c^2 = \alpha + \beta Z + \gamma Z^2 + |\delta|. \tag{1.40}$$

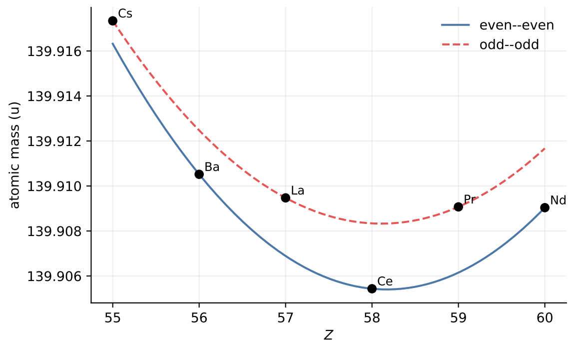

Figure 1.2: Even $A = 140$: pairing splits even–even (lower) and odd–odd (upper) chains. One β-stable even–even isobar near $Z = 58$.

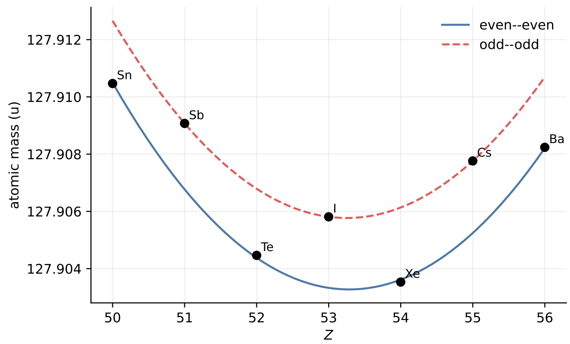

Figure 1.3: Even $A = 128$: two even–even isobars ($Z = 52, 54$) lie on the lower parabola; the odd–odd member can decay both ways. Some even–even chains are stable against *single* β but decay very slowly by double β (e.g. tellurium isotopes).

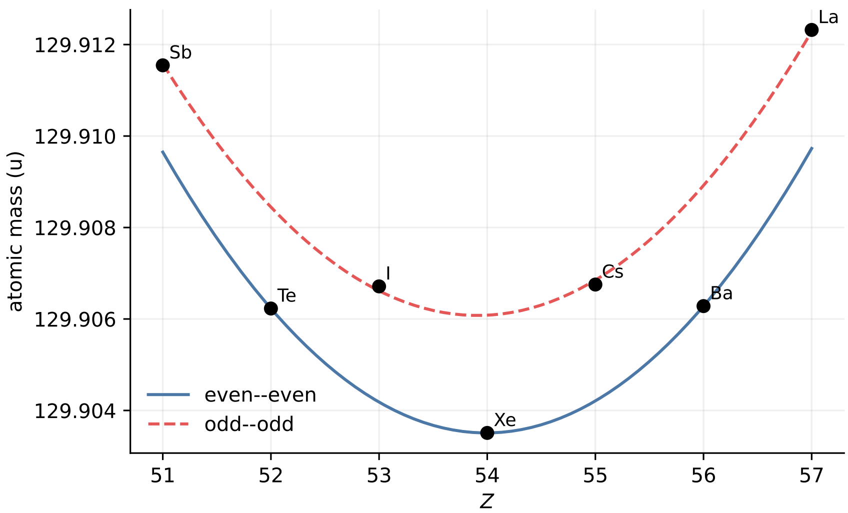

Figure 1.4: Even $A = 130$: three even–even nuclides ($Z = 52, 54, 56$) on the lower parabola can all be stable against single β decay relative to their odd–odd neighbours. Long-term evolution may still proceed by double β where $Q_{2\beta} > 0$.

1.5.1 Two effects absent from odd-A chains

1. **Bidirectional β decay of odd–odd nuclei.** An odd–odd isobar near the minimum of the *upper* parabola can have both neighbours on the lower parabola at lower mass. Then both β^- and β^+/EC are open. Classic examples: ^{128}I and ^{64}Cu [6, 1, 3].

2. **Double β decay of even–even nuclei.** If single β from an even–even nucleus would land on a higher odd–odd isobar, that single step is energetically forbidden, yet a transition to a lower even–even neighbour two units of Z away can still be allowed. With atomic masses,

$$Q_{2\beta^-} = [M(A,Z) - M(A,Z+2)]c^2. \tag{1.41}$$

   On the even–even parabola alone (pairing cancelled between parent and daughter), the same algebra that produced (1.38) with a unit step gives, for a double step,

$$Q_{2\beta^-} = -\beta - \gamma[(Z+2)^2 - Z^2] = -\beta - \gamma(4Z+4) = 2\gamma(Z_{\min} - Z - 1).$$

   The nucleus may then decay only by **double beta decay** ($2\nu\beta\beta$, or the still-unseen $0\nu\beta\beta$). Candidates include ^{76}Ge, ^{128}Te, ^{130}Te, and ^{136}Xe, where single β is blocked by pairing but $Q_{2\beta} > 0$ [6, 1, 4]. Observationally, such nuclei can be extremely long lived: ^{128}Te has a $2\nu\beta\beta$ half-life of order 10^{24} yr.

Key idea

Pairing does not invent new forces; it rearranges which isobars are allowed to β decay. Odd A: one parabola, one valley. Even A: two parabolas, possible double-β and two-way decays.

1.5.2 Why light odd–odd nuclei can be stable

For small A the parabola is narrow ($\gamma(A)$ is larger). The pairing offset can then place an odd–odd nucleus low enough to be stable against single β. The four stable odd–odd species ^{2}H, ^{6}Li, ^{10}B, ^{14}N live in that regime; heavier odd–odd nuclei almost always find an open β path to an even–even daughter [1, 6].

1.6 β Q values from the SEMF (worked structure)

Start again from the atomic ledger (1.3). The β^- mass difference is

$$\begin{aligned}[M(A,Z) - M(A,Z+1)]c^2 = {} & [Z\,m(^1\mathrm{H}) + (A-Z)\,m_n]c^2 - [(Z+1)\,m(^1\mathrm{H}) + (A-Z-1)\,m_n]c^2 \\ & - B(A,Z) + B(A,Z+1).\end{aligned} \tag{1.42}$$

The free-nucleon difference collapses to $[m_n - m(^1\mathrm{H})]c^2 = \delta_{nH}$, so

$$Q_{\beta^-}(A,Z) = \delta_{nH} + B(A,Z+1) - B(A,Z). \tag{1.43}$$

Insert the SEMF (1.1). Volume and surface cancel at fixed A. The remaining pieces are Coulomb, asymmetry, and pairing.

Write $\Delta X \equiv X(A,Z) - X(A,Z+1)$ for a term that enters M with a plus sign in (1.4). The Coulomb contribution is

$$\begin{aligned}\Delta_{\mathrm{C}} &= a_C A^{-1/3}\big[Z(Z-1) - (Z+1)Z\big] \\ &= a_C A^{-1/3}\big[Z^2 - Z - Z^2 - Z\big] = a_C A^{-1/3}(-2Z) = -2a_C A^{-1/3}\,Z. \end{aligned} \tag{1.44}$$

For asymmetry, set $u = A - 2Z$. Then the neighbour has $A - 2(Z+1) = u - 2$, and

$$\begin{aligned}(A-2Z)^2 - (A-2Z-2)^2 = u^2 - (u-2)^2 = u^2 - \big(u^2 - 4u + 4\big) = 4u - 4 \\ = 4(A-2Z) - 4 = 4(A-2Z-1). \end{aligned} \tag{1.45}$$

Therefore

$$\Delta_{\mathrm{a}} = \frac{a_{\mathrm{a}}}{A}\,4(A-2Z-1) = \frac{4a_{\mathrm{a}}}{A}(A-2Z-1). \tag{1.46}$$

For odd A (pairing zero),

$$Q_{\beta^-} = \delta_{nH} + \Delta_{\mathrm{C}} + \Delta_{\mathrm{a}}. \tag{1.47}$$

For even A, add the pairing difference $-\delta(A,Z) + \delta(A,Z+1)$. Moving toward $Z_{\min}$ lowers mass: Coulomb and asymmetry enter with opposite roles on the two sides of the valley.

Worked estimate: Q_{β^-} for ^{135}Cs

For odd $A = 135$, pairing vanishes and the SEMF gives a single parabola. With $a_C = 0.72$, $a_{\mathrm{a}} = 23$ (MeV), $Z_{\min} \approx 56$ so the valley sits near barium ($Z = 56$). Consider neutron-rich ^{135}Cs ($Z = 55$):

$$Q_{\beta^-} = \big[M(135,55) - M(135,56)\big]c^2.$$

Volume and surface cancel at fixed A. From (1.44)–(1.46) with $Z = 55$,

$$A^{1/3} = 135^{1/3} \approx 5.13, \qquad \Delta_{\mathrm{C}} = -2\cdot 0.72\cdot\frac{55}{5.13} = -0.72\cdot\frac{110}{5.13} \approx -15.4\,\mathrm{MeV},$$

$$A - 2Z = 135 - 110 = 25, \qquad \Delta_{\mathrm{a}} = \frac{4\cdot 23}{135}(25-1) = 23\cdot\frac{96}{135} \approx 16.36\,\mathrm{MeV}.$$

Altogether

$$Q_{\beta^-} \approx 0.782 - 15.44 + 16.36 \approx 1.70\,\mathrm{MeV}.$$

The evaluated atomic-mass difference gives only $Q_{\beta^-} \approx 0.269\,\mathrm{MeV}$ [27]. This is not a contradiction: ^{135}Cs lies very close to the parabola minimum, where a sub-MeV shell residual is comparable to the entire finite difference. The SEMF captures the direction and rough scale; evaluated masses, not the smooth formula, determine whether a near-valley decay is open and its precision Q value. The positive sign confirms β^- decay of neutron-rich ^{135}Cs toward ^{135}Ba.

Use evaluated masses for decay decisions

The continuous SEMF vertex answers "where is the valley approximately?" It does not certify an integer nuclide as stable. For neighbouring isobars compute

$$Q_{\beta^-} = [M_{\rm at}(A,Z) - M_{\rm at}(A,Z+1)]c^2$$

(and the corresponding β^+ or electron-capture expression) from an evaluated mass table such as AME2020. Pairing and shell corrections that are small in a total mass can dominate this near-cancellation [27].

Physics checklist for an isobar

1. Compute $Z_{\rm min}(A)$ from (1.31) (or (1.23)).
2. For odd A: one parabola; the integer Z nearest $Z_{\rm min}$ is stable; others β decay toward it with Q increasing up the sides via (1.38).
3. For even A: draw two parabolas split by $\sim 2|\delta|$; identify even–even minima and any odd–odd bidirectional cases; mark possible double-β pairs with (1.41).
4. Q_β follows from mass differences on that surface, with no weak matrix element required for the *sign* of the decay.

1.7 Connecting the valley of stability

Plotting the integer nearest to $Z_{\rm min}(A)$ for each A traces the valley of stability on the (N,Z) plane: $N \approx Z$ at low A, bending to $N > Z$ at high A. Nuclei below the valley (neutron excess) decay by β^-; nuclei above it decay by β^+ or electron capture. Lectures 4–5 therefore close the liquid-drop programme: binding energy and its Z landscape, including the β arrows that keep real nuclei near the valley. Lecture 6 follows the same $Z_{\rm min}(A)$ under electron degeneracy in neutron-star crusts; Lectures 18–19 turn the same Q values into spectral endpoints and ft values [6, 1, 13].

1.8 Problems with solutions

Problem 5.1 — Evaluate $Z_{\rm min}$

Using $Z_{\rm min} = A/[2(1+0.0078\,A^{2/3})]$, compute $Z_{\rm min}$ for $A = 100$ and $A = 200$. For each, give N/Z at that charge.

Solution 5.1

For $A = 100$, $A^{2/3} \approx 21.54$,

$$Z_{\rm min} = \frac{50}{1+0.0078\times 21.54} = \frac{50}{1.168} \approx 42.8 \quad \Rightarrow \quad \frac{N}{Z} \approx \frac{57.2}{42.8} \approx 1.34.$$

For $A = 200$, $A^{2/3} \approx 34.2$,

$$Z_{\min} = \frac{100}{1 + 0.0078 \times 34.2} = \frac{100}{1.267} \approx 78.9 \quad \Rightarrow \quad \frac{N}{Z} \approx \frac{121}{79} \approx 1.53.$$

Heavy nuclei are neutron rich [6].

Problem 5.2 — Free-mass difference

Show that $(m_n - m(^1\mathrm{H}))c^2 \approx 0.782\,\mathrm{MeV}$ using $m_n = 1.008665\,\mathrm{u}$, $m(^1\mathrm{H}) = 1.007825\,\mathrm{u}$, and $c^2 = 931.5\,\mathrm{MeV/u}$. Where does this number enter $Z_{\min}$?

Solution 5.2

$$(1.008665 - 1.007825) \times 931.5 = 0.000840 \times 931.5 \approx 0.782\,\mathrm{MeV}.$$

It appears in the numerator of the exact $Z_{\min}$ formula (1.23) as the free-mass drive toward protons; it is small compared with $4a_{\mathrm{a}} \approx 92\,\mathrm{MeV}$ [6, 14].

Problem 5.3 — Q_{β^-} sign on the parabola

At fixed odd A, $M(Z)$ is a parabola with minimum at $Z_{\min}$. Show that $Q_{\beta^-} = [M(Z) - M(Z+1)]c^2 > 0$ for all integers $Z < Z_{\min} - 1/2$, so neutron-rich isobars β^- decay toward the minimum.

Solution 5.3

Use the chapter convention $Mc^2 = \alpha + \beta Z + \gamma Z^2$ ($\delta = 0$) with $\gamma > 0$. Then

$$[M(Z) - M(Z+1)]c^2 = -\beta - \gamma(2Z+1).$$

The minimum is at $Z_{\min} = -\beta/(2\gamma)$ (1.14), so $-\beta = 2\gamma Z_{\min}$ and

$$Q_{\beta^-} = 2\gamma\big(Z_{\min} - Z - \tfrac{1}{2}\big),$$

as in (1.38). For integer $Z \le Z_{\min} - 1$ this is positive: $Q_{\beta^-} > 0$. Geometry alone forces β^- on the neutron-rich side [6, 1].

Problem 5.4 — Pairing split and double β

Even–even and odd–odd mass parabolas at fixed even A are split by $2|\delta|$ with $|\delta| = a_P A^{-3/4}$. For $A = 128$ and $a_P = 34\,\mathrm{MeV}$, estimate the vertical split $2|\delta|$. Explain in one sentence why an even–even nucleus can be stable against single β yet unstable against double β.

Solution 5.4

$$A^{3/4} = 128^{0.75} \approx (2^7)^{0.75} = 2^{5.25} \approx 38, \quad |\delta| \approx 34/38 \approx 0.9\,\text{MeV}, \quad 2|\delta| \approx 1.8\,\text{MeV}.$$

Single β would land on the higher odd–odd parabola, so $Q_\beta < 0$ for that step, while a jump of $\Delta Z = 2$ to a lower even–even neighbour can still have $Q_{2\beta} > 0$ [6, 1].

Problem 5.5 — Both β modes (Petrera-style)

For odd–odd ^{64}Cu near mid-mass, explain qualitatively why both β^- (to ^{64}Zn) and β^+ (to ^{64}Ni) can be open, using the two-parabola picture.

Solution 5.5

^{64}Cu sits on the upper (odd–odd) parabola. Both even–even neighbours ^{64}Ni and ^{64}Zn lie on the lower parabola and can be lower in mass than ^{64}Cu. Then both Q_{β^-} and Q_{β^+} are positive: bidirectional decay. Quantitative SEMF estimates give $Q \sim 1\,\text{MeV}$ for both branches [3, 6].

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